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Matrix models and D-branes in twistor string theory

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Abstract: We construct two matrix models from twistor string theory: one by dimensional reduction onto a rational curve and another one by introducing noncommutative coordinates on the fibres of the supertwistor space $\mathcal{P}^{3|4} \to \mathbb{C}P^1$. We comment on the interpretation of our matrix models in terms of topological D-branes and relate them to a recently proposed string field theory. By extending one of the models, we can carry over all the ingredients of the super ADHM construction to a D-brane configuration in the supertwistor space $\mathcal{P}^{3|4}$. Eventually, we present the analogue picture for the (super) Nahm construction.

Keywords: D-branes, Superstrings and Heterotic Strings, Integrable Field Theories, Matrix Models.

Contents

1. Introduction

The basic idea of twistor string theory [1][1] is the union of twistor geometry with Calabi-Yau geometry in the supermanifold $\mathbb{C}P^{3|4}$. This space is simultaneously a supertwistor space and a Calabi-Yau supermanifold and one can use it as a target space for a topological B-model, which can be shown to be equivalent to $\mathcal{N} = 4$ supersymmetrically extended self-dual Yang-Mills (SDYM) theory. By incorporating additional D-instantons into the picture, one can obtain the full $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory and use its twistorial and string theoretical description for calculating amplitudes in this theory. For a good overview of the results in this area, see e.g. [3].

Already a number of variations and reductions of the underlying supertwistor space $\mathbb{C}P^{3|4}$ and its open subset $\mathcal{P}^{3|4} = \mathbb{C}P^{3|4}\backslash\mathbb{C}P^{1|4}$ have been considered [4−13]. In this paper, we want to discuss dimensional reductions of the bosonic dimensions of $\mathcal{P}^{3|4}$ and construct matrix models from the twistor string. For obtaining these, we will use two methods. Starting point of both is holomorphic Chern-Simons (hCS) theory on the noncompact supertwistor space $\mathcal{P}^{3|4}$, which is a rank $2|4$ (complex) vector bundle over the Riemann sphere $\mathbb{C}P^1$ and becomes diffeomorphic to $\mathbb{R}^{4|8} \times \mathbb{C}P^1$ after imposing reality conditions on its sections. Here, $\mathbb{R}^{4|8}$ is the moduli space of (real) holomorphic sections of $\mathcal{P}^{3|4}$.

In the first approach, we will dimensionally reduce $\mathcal{P}^{3|4}$ to the rank $0|4$ vector bundle $\mathbb{C}P^{1|4}$ over $\mathbb{C}P^1$. Via the twistor correspondence, this amounts to reducing the moduli space $\mathbb{R}^{4|8}$ by its bosonic coordinates to $\mathbb{R}^{0|8}$: $\mathbb{C}P^{1|4} \cong \mathbb{R}^{0|8} \times \mathbb{C}P^1$. On the field theory side, we will obtain an action corresponding to matrix quantum mechanics with complex time over $\mathbb{C}P^{1|4}$. For a similar construction on the conifold, see [14].

The second method will be to impose a noncommutative algebra on the bosonic coordinates of the moduli space $\mathbb{R}^{4|8}$, which yields a noncommutative algebra for the fibre coordinates of $\mathcal{P}^{3|4}$. This turns the derivatives and its coordinates into operators in an infinite dimensional Fock space, which can be represented by infinite dimensional matrices. In this sense, hCS theory will again be reduced to matrix quantum mechanics, as the integral over the bosonic moduli becomes a trace over the Fock space.

Starting from hCS theory with gauge group $\mathrm{GL}(n, \mathbb{C})$, the first method yields a matrix model whose field content takes values in the Lie algebra of $\mathrm{GL}(n, \mathbb{C})$. One expects this model to be equivalent to the second one in an appropriate limit $n \to \infty$. Furthermore, both models can be reduced by integrating over the remaining bosonic coordinate of $\mathbb{C}P^{1|4}$, which leads to matrix models of $\mathcal{N} = 4$ SDYM theory.

Having defined these matrix models, we will elaborate on their physical interpretation and discuss their relation to the cubic string field theory proposed in [15] as well as their rôles as effective actions for certain D-brane configurations.

The D-brane interpretation of the matrix models will always be twofold: On the one hand, we have physical D-branes in type IIB superstring theory with the moduli space $\mathbb{R}^{4|8}$ being a subspace of the ten-dimensional target space. On the other hand, we have topological D-branes of B-type topological string theory in the supertwistor space $\mathcal{P}^{3|4}$. By

[1] See [2] for an alternative formulation.

extending the matrix model on $\mathbb{C}P^{1|4}$, we will be able to carry over all the ingredients of the D-brane interpretation of the ADHM construction from the moduli space $\mathbb{R}^{4|8}$ to the supertwistor space $\mathcal{P}^{3|4}$. Introducing further dimensional reductions of hCS theory on $\mathcal{P}^{3|4}$, we do the same for the D-brane configuration describing the super Nahm construction.

Together with [16], this paper is intended as a first step towards a D-brane interpretation of solutions to noncommutative self-dual Yang-Mills theory and dimensional reductions thereof [17].

The outline of this paper is as follows. Section 2 is devoted to a review of the geometry of $\mathcal{P}^{3|4}$, hCS theory on this space and this theory's relation to $\mathcal{N}=4$ SDYM theory on $\mathbb{R}^{4|8}$. In section 3, we construct the matrix models from both dimensional reduction and noncommutativity. Furthermore, we point out the similarity of the matrix models with a string field theory. The D-brane interpretation of the matrix models and the extension to an ADHM model in $\mathcal{P}^{3|4}$ is presented in section 4. The corresponding picture for the Nahm construction is drawn in section 5, and we conclude in section 6.

2. Holomorphic Chern-Simons theory on $\mathcal{P}^{3|4}$

Recall that the open topological B-model on a complex three-dimensional Calabi-Yau manifold with a stack of n D5-branes is equivalent to holomorphic Chern-Simons (hCS) theory which describes holomorphic structures on a rank n vector bundle over the same space [1]. In this section, we will briefly review the definitions of $\mathcal{N}=4$ supersymetrically extended hCS theory on the supertwistor space $\mathcal{P}^{3|4} = \mathbb{C}P^{3|4}\backslash\mathbb{C}P^{1|4}$ arising from the topological B-model on $\mathcal{P}^{3|4}$. For a more detailed discussion, see e.g. [1, 18].

2.1 The complex twistor correspondence

Consider the Riemann sphere $\mathbb{C}P^1 \cong S^2$ with complex homogeneous coordinates $\lambda_{\dot 1}$ and $\lambda_{\dot 2}$. We can cover this space by two patches U_+ and U_- for which $\lambda_{\dot 1} \neq 0$ and $\lambda_{\dot 2} \neq 0$, respectively, and introduce the standard (affine) complex coordinates $\lambda_+ := \frac{\lambda_{\dot 2}}{\lambda_{\dot 1}}$ on U_+ and $\lambda_- := \frac{\lambda_{\dot 1}}{\lambda_{\dot 2}}$ on U_- with $\lambda_+ = (\lambda_-)^{-1}$ on the overlap $U_+ \cap U_-$.

The sections of the holomorphic line bundle $\mathcal{O}(1)$ over $\mathbb{C}P^1$ are described by holomorphic functions $z_\pm$ over $U_\pm$ which are related by $z_+ = \lambda_+ z_-$ on the intersection $U_+ \cap U_-$. Using the parity changing operator Π, which inverts the parity of the fibre coordinates when acting on a fibre bundle, we also define the bundle $\Pi\mathcal{O}(1)$ whose sections are described by holomorphic Graßmann-valued functions $\eta_\pm$ over $U_\pm$ with $\eta_+ = \lambda_+\eta_-$ on $U_+ \cap U_-$.

We can now define the supertwistor space

$$\mathcal{P}^{3|4} \; := \; \mathbb{C}P^{3|4}\backslash\mathbb{C}P^{1|4} \; = \; \mathbb{C}^2 \otimes \mathcal{O}(1) \oplus \mathbb{C}^4 \otimes \Pi\mathcal{O}(1) \tag{2.1}$$

as the total space of a rank $2|4$ holomorphic vector bundle

$$\mathcal{P}^{3|4} \;\; \rightarrow \;\; \mathbb{C}P^1 \; . \tag{2.2}$$

This bundle can be covered by the two patches

$$\mathcal{U}_+ \; := \; \mathcal{P}^{3|4}\Big|_{U_+} \; \cong \; U_+ \times \mathbb{C}_+^{2|4} \quad \text{and} \quad \mathcal{U}_- \; := \; \mathcal{P}^{3|4}\Big|_{U_-} \; \cong \; U_- \times \mathbb{C}_-^{2|4} \tag{2.3}$$

with coordinates $(z^\alpha_\pm, \lambda_\pm, \eta^\pm_i)$, where $\lambda_\pm$ are the coordinates on $U_\pm$, $z^\alpha_\pm$ with $\alpha = 1,2$ are the coordinates on the bosonic fibres $\mathbb{C}^{2|0}_\pm$ and $\eta^\pm_i$ with $i = 1,\ldots,4$ are the coordinates on the fermionic fibres $\mathbb{C}^{0|4}_\pm$. For convenience, we also introduce $z^3_\pm = \lambda_\pm$.

Consider now the complex superspace $\mathbb{C}^{4|8}$ with coordinates[2]

$$x^{\alpha\dot\alpha} \quad \text{on} \quad \mathbb{C}^{4|0} \quad \text{and} \quad \eta^{\dot\alpha}_i \quad \text{on} \quad \mathbb{C}^{0|8}\ , \quad \alpha,\dot\alpha \;=\; 1,2\ , \tag{2.4}$$

and the so-called *correspondence space*

$$\mathcal{F}^{5|8} \;:=\; \mathbb{C}^{4|8}\times\mathbb{C}P^1\ . \tag{2.5}$$

We can define a projection $\pi_2 : \mathcal{F}^{5|8} \to \mathcal{P}^{3|4}$ by the formula

$$\pi_2(x^{\alpha\dot\alpha}, \lambda^\pm_{\dot\alpha}, \eta^{\dot\alpha}_i) \;:=\; (z^\alpha_\pm, \lambda_\pm, \eta^\pm_i) \tag{2.6}$$

with

$$z^\alpha_\pm \;:=\; x^{\alpha\dot\alpha}\lambda^\pm_{\dot\alpha} \quad \text{and} \quad \eta^\pm_i \;:=\; \eta^{\dot\alpha}_i\lambda^\pm_{\dot\alpha}\ , \tag{2.7}$$

where

$$(\lambda^+_{\dot\alpha}) \;:=\; \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} \quad \text{and} \quad (\lambda^-_{\dot\alpha}) \;:=\; \begin{pmatrix} \lambda_- \\ 1 \end{pmatrix}\ . \tag{2.8}$$

Note that formulæ (2.7), the *incidence relations*, with fixed $x^{\alpha\dot\alpha}, \eta^i_{\dot\alpha}$ define holomorphic sections of the bundle (2.2) which are projective lines $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|4}$.

There is also the canonical projection $\pi_1 : \mathcal{F}^{5|8} \to \mathbb{C}^{4|8}$ given explicitly by the formula

$$\pi_1(x^{\alpha\dot\alpha}, \lambda^\pm_{\dot\alpha}, \eta^{\dot\alpha}_i) \;:=\; (x^{\alpha\dot\alpha}, \eta^{\dot\alpha}_i)\ . \tag{2.9}$$

Together, the two projections π_1 and π_2 define the double fibration

$$\begin{tikzcd} & \mathcal{F}^{5|8} \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ \mathcal{P}^{3|4} & & \mathbb{C}^{4|8} \end{tikzcd} \tag{2.10}$$

which yields a *twistor correspondence* between $\mathcal{P}^{3|4}$ and $\mathbb{C}^{4|8}$, i.e. a correspondence between points in one space and subspaces of the other one:

$$\begin{gathered} \big\{\,\text{points } p \text{ in } \mathcal{P}^{3|4}\big\} \;\longleftrightarrow\; \big\{\,\text{null } (2|4)\text{-dimensional superplanes } \pi_1(\pi_2^{-1}(p)) \text{ in } \mathbb{C}^{4|8}\big\}\ , \\ \big\{\,\text{projective lines } \mathbb{C}P^1_{x,\eta} \;=\; \pi_2(\pi_1^{-1}(x,\eta)) \hookrightarrow \mathcal{P}^{3|4}\big\} \;\longleftrightarrow\; \big\{\,\text{points } (x,\eta) \text{ in } \mathbb{C}^{4|8}\big\}\ . \end{gathered}$$

2.2 The real twistor correspondence

Recall that a real structure on a complex manifold X is defined as an anti-linear involution $\tau : X \to X$. On $\mathbb{C}^{4|8}$, we can define the following ones:

$$\begin{gathered} \tau_\varepsilon(x^{1\dot1}) \;=\; \bar x^{2\dot2}\ , \quad \tau_\varepsilon(x^{1\dot2}) \;=\; \varepsilon\bar x^{2\dot1} \quad \text{with} \quad \varepsilon \;=\; \pm1\ , \\ \tau_{+1}(\eta^{\dot1}_i) \;=\; \bar\eta^{\dot2}_i \quad \text{and} \quad \tau_{-1}\begin{pmatrix} \eta^{\dot1}_1 & \eta^{\dot1}_2 & \eta^{\dot1}_3 & \eta^{\dot1}_4 \\ \eta^{\dot2}_1 & \eta^{\dot2}_2 & \eta^{\dot2}_3 & \eta^{\dot2}_4 \end{pmatrix} \;=\; \begin{pmatrix} -\bar\eta^{\dot2}_2 & \bar\eta^{\dot2}_1 & -\bar\eta^{\dot2}_4 & \bar\eta^{\dot2}_3 \\ \bar\eta^{\dot1}_2 & -\bar\eta^{\dot1}_1 & \bar\eta^{\dot1}_4 & -\bar\eta^{\dot1}_3 \end{pmatrix}\ . \end{gathered} \tag{2.11}$$

[2] In fact, this space is the (anti-)chiral superspace with half of the fermionic coordinates of the full $\mathcal{N}=4$ superspace $\mathbb{C}^{4|16}$, see e.g. [18] for more details.

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(A third possible involution τ_0 is given e.g. in [18].) The corresponding reality conditions are then obtained by demanding invariance under the maps τ_ε, i.e. to impose the conditions

$$x^{1\dot{1}} \;=\; \bar{x}^{2\dot{2}} \;, \quad x^{1\dot{2}} \;=\; \varepsilon\bar{x}^{2\dot{1}} \;, \quad \begin{cases} \qquad \eta_i^{\dot{1}} \;=\; \bar{\eta}_i^{\dot{2}} & \text{for } \varepsilon \;=\; +1 \\ \eta_1^{\dot{1}} \;=\; -\bar{\eta}_2^{\dot{2}} \;, \; \eta_1^{\dot{2}} \;=\; \bar{\eta}_2^{\dot{1}} & \text{for } \varepsilon \;=\; -1 \;. \\ \eta_3^{\dot{1}} \;=\; -\bar{\eta}_4^{\dot{2}} \;, \; \eta_3^{\dot{2}} \;=\; \bar{\eta}_4^{\dot{1}} & \end{cases} \tag{2.12}$$

We thus obtain the real superspace $\mathbb{R}^{4|8} \subset \mathbb{C}^{4|8}$ together with a natural metric on its body defined by $\mathrm{d}s^2 = \det(\mathrm{d}x^{\alpha\dot{\alpha}})$. This metric is of Kleinian signature $(--++)$ for $\varepsilon = +1$ and of Euclidean signature $(++++)$ for $\varepsilon = -1$. The literal identification with the coordinates x^μ on $\mathbb{R}^4$ is then chosen to be

$$x^{2\dot{2}} \;=\; \bar{x}^{1\dot{1}} \;=:\; -(\varepsilon x^4 + \mathrm{i}x^3) \quad \text{and} \quad x^{2\dot{1}} \;=\; \varepsilon\bar{x}^{1\dot{2}} \;=:\; -\varepsilon(x^2 - \mathrm{i}x^1) \;. \tag{2.13}$$

Defining the anti-holomorphic involution on $\mathbb{C}P^1$ as

$$\tau_\varepsilon(\lambda_\pm) \;=\; \frac{\varepsilon}{\bar{\lambda}_\pm} \tag{2.14}$$

and using the incidence relations (2.7), we obtain

$$\tau_\varepsilon(z_+^1, z_+^2, \lambda_+) \;=\; \left(\frac{\bar{z}_+^2}{\bar{\lambda}_+}, \frac{\varepsilon\bar{z}_+^1}{\bar{\lambda}_+}, \frac{\varepsilon}{\bar{\lambda}_+}\right) \quad \text{and} \quad \tau_\varepsilon(z_-^1, z_-^2, \lambda_-) \;=\; \left(\frac{\varepsilon\bar{z}_-^2}{\bar{\lambda}_-}, \frac{\bar{z}_-^1}{\bar{\lambda}_-}, \frac{\varepsilon}{\bar{\lambda}_-}\right) \tag{2.15}$$

on the bosonic coordinates of the supertwistor space $\mathcal{P}^{3|4}$ and

$$\tau_1(\eta_i^\pm) \;=\; \left(\frac{\bar{\eta}_i^\pm}{\bar{\lambda}_\pm}\right) \;, \quad \tau_{-1}(\eta_1^\pm, \eta_2^\pm, \eta_3^\pm, \eta_4^\pm) \;=\; \left(\frac{\mp\bar{\eta}_2^\pm}{\bar{\lambda}_\pm}, \frac{\pm\bar{\eta}_1^\pm}{\bar{\lambda}_\pm}, \frac{\mp\bar{\eta}_4^\pm}{\bar{\lambda}_\pm}, \frac{\pm\bar{\eta}_3^\pm}{\bar{\lambda}_\pm}\right) \tag{2.16}$$

on its fermionic coordinates. It is obvious from (2.15) and (2.16) that the involution τ_{-1} has no fixed points, but does leave invariant projective lines $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|4}$ with $(x, \eta) \in \mathbb{R}^{4|8}$. On the other hand, the involution τ_1 does have fixed points which form a real supermanifold $\mathcal{T}^{3|4}$ with coordinates $(z_\pm^\alpha, \lambda_\pm, \eta_i^\pm)$ satisfying the reality conditions $\tau_1(z_\pm^\alpha, \lambda_\pm, \eta_i^\pm) = (z_\pm^\alpha, \lambda_\pm, \eta_i^\pm)$, i.e.

$$z_\pm^2 \;=\; \lambda_\pm\bar{z}_\pm^1 \;, \quad \lambda_\pm\bar{\lambda}_\pm \;=\; 1 \;, \quad \eta_i^\pm \;=\; \lambda_\pm\bar{\eta}_i^\pm \;. \tag{2.17}$$

For the space $\mathbb{R}^{4|8}$, we introduce a new correspondence space $\mathbb{R}^{4|8} \times \mathbb{C}P^1$ with the same projections (2.7) onto $\mathcal{P}^{3|4}$ and (2.9) onto $\mathbb{R}^{4|8}$ as well as the double fibration

$$\begin{tikzcd} & \mathbb{R}^{4|8} \times \mathbb{C}P^1 \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ \mathcal{P}^{3|4} & & \mathbb{R}^{4|8} \end{tikzcd} \tag{2.18}$$

This diagram describes very different situations in the Euclidean and the Kleinian case. For $\varepsilon = -1$, the map π_2 is a diffeomorphism,

$$\mathcal{P}^{3|4} \;\cong\; \mathbb{R}^{4|8} \times \mathbb{C}P^1 \;, \tag{2.19}$$

and the double fibration (2.18) is simplified to the non-holomorphic fibration

$$\pi_1 : \; \mathcal{P}^{3|4} \;\to\; \mathbb{R}^{4|8} \tag{2.20}$$

where 3|4 stands for complex and 4|8 for real dimensions. Correspondingly, we can choose either coordinates $(z^\alpha_\pm, z^3_\pm := \lambda_\pm, \eta^\pm_i)$ or $(x^{\alpha\dot\alpha}, \lambda_\pm, \eta^\pm_i)$ on $\mathcal{P}^{3|4}$ and consider this space as a complex 3|4-dimensional or a real 6|8-dimensional manifold.

In the case of Kleinian signature $(--++)$, we have local isomorphisms

$$\mathrm{SO}(2,2) \;\cong\; \mathrm{Spin}(2,2) \;\cong\; \mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R}) \;\cong\; \mathrm{SU}(1,1)\times\mathrm{SU}(1,1) \tag{2.21}$$

and under the action of the group $\mathrm{SU}(1,1)$, the Riemann sphere $\mathbb{C}P^1$ of projective spinors decomposes into the disjoint union $\mathbb{C}P^1 = H^2_+ \cup S^1 \cup H^2_- = H^2 \cup S^1$ of three orbits. Here, $H^2 = H^2_+ \cup H^2_-$ is the two-sheeted hyperboloid and $H^2_\pm = \{\lambda_\pm \in U_\pm \,|\, |\lambda_\pm| < 1\} \cong \mathrm{SU}(1,1)/\mathrm{U}(1)$ are open discs. This induces a decomposition of the correspondence space into

$$\mathbb{R}^{4|8}\times\mathbb{C}P^1 \;=\; \mathbb{R}^{4|8}\times H^2_+ \cup \mathbb{R}^{4|8}\times S^1 \cup \mathbb{R}^{4|8}\times H^2_- \;=\; \mathbb{R}^{4|8}\times H^2 \cup \mathbb{R}^{4|8}\times S^1 \tag{2.22}$$

as well as a decomposition of the twistor space

$$\mathcal{P}^{3|4} \;=\; \mathcal{P}^{3|4}_+ \cup \mathcal{P}_0 \cup \mathcal{P}^{3|4}_- \;=:\; \tilde{\mathcal{P}}^{3|4} \cup \mathcal{P}_0\ , \tag{2.23}$$

where $\mathcal{P}^{3|4}_\pm := \mathcal{P}^{3|4}|_{H^2_\pm}$ are restrictions of the rank 2|4 holomorphic vector bundle (2.2) to bundles over $H^2_\pm$. The space $\mathcal{P}_0 := \mathcal{P}^{3|4}|_{S^1}$ is the real 5|8-dimensional common boundary of the spaces $\mathcal{P}^{3|4}_\pm$. There is a natural map from $\mathbb{R}^{4|8}\times H^2$ into $\tilde{\mathcal{P}}^{3|4}$ which is a diffeomorphism between $\mathbb{R}^{4|8}\times H^2_\pm$ and $\mathcal{P}^{3|4}_\pm$,

$$\tilde{\mathcal{P}}^{3|4} \;\cong\; \mathbb{R}^{4|8}\times H^2 \tag{2.24}$$

given again by formulæ (2.7) and their inverses. Thus, we have a fibration analogously to (2.20),

$$\tilde{\mathcal{P}}^{3|4} \;\to\; \mathbb{R}^{4|8}\ . \tag{2.25}$$

On the space $\mathbb{R}^{4|8}\times S^1 \subset \mathbb{R}^{4|8}\times\mathbb{C}P^1$, the map (2.7) becomes a real fibration

$$\mathbb{R}^{4|8}\times S^1 \;\to\; \mathcal{T}^{3|4} \hookrightarrow \mathcal{P}_0 \tag{2.26}$$

over the real 3|4-dimensional space. That is why for (p,q) forms, vector fields of type $(0,1)$ etc., we should consider the spaces (2.24) and the fibration (2.25). However, holomorphic vector bundles which are described by solutions of hCS theory on $\tilde{\mathcal{P}}^{3|4} \subset \mathcal{P}^{3|4}$ can be extended to bundles over the whole twistor space. For more details on this, see e.g. [18, 11]. To indicate which spaces we are working with, we will use the notation $\mathcal{P}^{3|4}_\varepsilon$ and imply $\mathcal{P}^{3|4}_{-1} := \mathcal{P}^{3|4}$ and $\mathcal{P}^{3|4}_{+1} := \tilde{\mathcal{P}}^{3|4} \subset \mathcal{P}^{3|4}$.

2.3 Antiholomorphic vector fields on $\mathcal{P}^{3|4}_\varepsilon$

On $\mathcal{P}^{3|4}_\varepsilon$, there is the following relationship between vector fields of type (0,1) in the coordinates $(z^\alpha_\pm, z^3_\pm, \eta^\pm_i)$ and vector fields in the coordinates $(x^{\alpha\dot\alpha}, \lambda_\pm, \eta^{\dot\alpha}_i)$:

$$\begin{aligned}
\frac{\partial}{\partial\bar z^1_\pm} &= -\gamma_\pm\lambda^{\dot\alpha}_\pm\frac{\partial}{\partial x^{2\dot\alpha}} =: -\gamma_\pm\bar V^\pm_2\ , &\qquad \frac{\partial}{\partial\bar z^2_\pm} &= \gamma_\pm\lambda^{\dot\alpha}_\pm\frac{\partial}{\partial x^{1\dot\alpha}} =: -\varepsilon\gamma_\pm\bar V^\pm_1\ ,\\
\frac{\partial}{\partial\bar z^3_+} &= \frac{\partial}{\partial\bar\lambda_+} + \varepsilon\gamma_+ x^{\alpha\dot 1}\bar V^+_\alpha + \varepsilon\gamma_+\eta^{\dot 1}_i\bar V^i_+\ , &\qquad \frac{\partial}{\partial\bar z^3_-} &= \frac{\partial}{\partial\bar\lambda_-} + \gamma_- x^{\alpha\dot 2}\bar V^-_\alpha + \gamma_-\eta^{\dot 2}_i\bar V^i_-\ .
\end{aligned} \tag{2.27}$$

In the Kleinian case, one obtains for the fermionic vector fields

$$\frac{\partial}{\partial \bar{\eta}_i^{\pm}} = -\gamma_{\pm} \bar{V}_{\pm}^i := -\gamma_{\pm} \lambda_{\pm}^{\dot{\alpha}} \frac{\partial}{\partial \eta_i^{\dot{\alpha}}} , \tag{2.28}$$

while in the Euclidean case, we have

$$\begin{aligned} \frac{\partial}{\partial \bar{\eta}_1^{\pm}} &= \gamma_{\pm} \lambda_{\pm}^{\dot{\alpha}} \frac{\partial}{\partial \eta_2^{\dot{\alpha}}} =: \gamma_{\pm} \bar{V}_{\pm}^2 , \quad \frac{\partial}{\partial \bar{\eta}_2^{\pm}} = -\gamma_{\pm} \lambda_{\pm}^{\dot{\alpha}} \frac{\partial}{\partial \eta_1^{\dot{\alpha}}} =: -\gamma_{\pm} \bar{V}_{\pm}^1 , \\ \frac{\partial}{\partial \bar{\eta}_3^{\pm}} &= \gamma_{\pm} \lambda_{\pm}^{\dot{\alpha}} \frac{\partial}{\partial \eta_4^{\dot{\alpha}}} =: \gamma_{\pm} \bar{V}_{\pm}^4 , \quad \frac{\partial}{\partial \bar{\eta}_4^{\pm}} = -\gamma_{\pm} \lambda_{\pm}^{\dot{\alpha}} \frac{\partial}{\partial \eta_3^{\dot{\alpha}}} =: -\gamma_{\pm} \bar{V}_{\pm}^3 . \end{aligned} \tag{2.29}$$

All these relations follow from the formulæ (2.7) and their inverses. In the above equations, we introduced the factors

$$\gamma_+ = \frac{1}{1 - \varepsilon \lambda_+ \bar{\lambda}_+} = \frac{1}{\hat{\lambda}_+^{\dot{\alpha}} \lambda_{\dot{\alpha}}^+} \quad \text{and} \quad \gamma_- = -\varepsilon \frac{1}{1 - \varepsilon \lambda_- \bar{\lambda}_-} = \frac{1}{\hat{\lambda}_-^{\dot{\alpha}} \lambda_{\dot{\alpha}}^-} , \tag{2.30}$$

where indices are raised and lowered as usual with the antisymmetric tensor of $\mathrm{SL}(2, \mathbb{C})$. For the latter, we use the convention $\varepsilon^{\dot{1}\dot{2}} = -\varepsilon_{\dot{1}\dot{2}} = 1$ (which implies that $\varepsilon_{\dot{\alpha}\dot{\beta}} \varepsilon^{\dot{\beta}\dot{\gamma}} = \delta_{\dot{\alpha}}^{\dot{\gamma}}$). The coordinates $\hat{\lambda}_{\dot{\alpha}}^{\pm}$ are obtained from the coordinates $\lambda_{\dot{\alpha}}^{\pm}$ by an appropriate action of the real structure τ_ε [18]. To obtain the coordinates $\hat{\lambda}_{\pm}^{\dot{\alpha}}$, one first raises the index and then applies the action of τ. Altogether, we have the following variants of the two-spinor $\lambda_{\dot{\alpha}}^{\pm}$:

$$(\lambda_+^{\dot{\alpha}}) := (\varepsilon^{\dot{\alpha}\dot{\beta}} \lambda_{\dot{\beta}}^+) = \begin{pmatrix} \lambda_+ \\ -1 \end{pmatrix} , \quad (\lambda_-^{\dot{\alpha}}) := \begin{pmatrix} 1 \\ -\lambda_- \end{pmatrix} , \tag{2.31}$$

$$(\hat{\lambda}_{\dot{\alpha}}^+) := \begin{pmatrix} \varepsilon \bar{\lambda}_+ \\ 1 \end{pmatrix} , \quad (\hat{\lambda}_{\dot{\alpha}}^-) := \begin{pmatrix} \varepsilon \\ \bar{\lambda}_- \end{pmatrix} , \quad (\hat{\lambda}_+^{\dot{\alpha}}) := \begin{pmatrix} -\varepsilon \\ \bar{\lambda}_+ \end{pmatrix} , \quad (\hat{\lambda}_-^{\dot{\alpha}}) := \begin{pmatrix} -\varepsilon \bar{\lambda}_- \\ 1 \end{pmatrix} .$$

2.4 Forms on $\mathcal{P}_\varepsilon^{3|4}$

One can introduce the (nowhere vanishing) holomorphic volume form

$$\Omega_{\pm} := \Omega|_{\mathcal{U}_{\pm}} := \pm \mathrm{d}\lambda_{\pm} \wedge \mathrm{d}z_{\pm}^1 \wedge \mathrm{d}z_{\pm}^2 \mathrm{d}\eta_1^{\pm} \dots \mathrm{d}\eta_4^{\pm} =: \pm \mathrm{d}\lambda_{\pm} \wedge \mathrm{d}z_{\pm}^1 \wedge \mathrm{d}z_{\pm}^2 \, \Omega_{\pm}^{\eta} \tag{2.32}$$

on $\mathcal{P}^{3|4}$. The existence of this volume element implies that the Berezinian line bundle is trivial and consequently $\mathcal{P}^{3|4}$ is a Calabi-Yau supermanifold [1]. Note, however, that Ω is not a differential form because its fermionic part transforms as a product of Graßmann-odd vector fields, i.e. with the inverse of the Jacobian. Such forms are called integral forms.

It will also be useful to introduce $(0, 1)$-forms $\bar{E}_{\pm}^a$ and $\bar{E}_i^{\pm}$ which are dual to $\bar{V}_a^{\pm}$ and $\bar{V}_{\pm}^i$, respectively, i.e.

$$\bar{V}_a^{\pm} \lrcorner \bar{E}_{\pm}^b = \delta_a^b \quad \text{and} \quad \bar{V}_{\pm}^i \lrcorner \bar{E}_j^{\pm} = \delta_j^i . \tag{2.33}$$

Here, $\lrcorner$ denotes the interior product of vector fields with differential forms. Explicitly, the dual $(0, 1)$-forms are given by the formulæ

$$\bar{E}_{\pm}^{\alpha} = -\gamma_{\pm} \hat{\lambda}_{\dot{\alpha}}^{\pm} \mathrm{d}x^{\alpha\dot{\alpha}} , \quad \bar{E}_{\pm}^3 = \mathrm{d}\bar{\lambda}_{\pm} \quad \text{and} \quad \bar{E}_i^{\pm} = -\gamma_{\pm} \hat{\lambda}_{\dot{\alpha}}^{\pm} \mathrm{d}\eta_i^{\dot{\alpha}} . \tag{2.34}$$

2.5 Holomorphically embedded submanifolds and their normal bundles

Equations (2.7) describe a holomorphic embedding of the space $\mathbb{C}P^1$ into the supertwistor space $\mathcal{P}^{3|4}$. That is, for fixed moduli $x^{\alpha\dot{\alpha}}$ and $\eta_i^{\dot{\alpha}}$, equations (2.7) yield a projective line $\mathbb{C}P^1_{x,\eta}$ inside the supertwistor space. The normal bundle to any $\mathbb{C}P^1_{x,\eta}\hookrightarrow\mathcal{P}^{3|4}$ is $\mathcal{N}^{2|4} = \mathbb{C}^2\otimes\mathcal{O}(1)\oplus\mathbb{C}^4\otimes\Pi\mathcal{O}(1)$ and we have

$$h^0(\mathbb{C}P^1_{x,\eta},\mathcal{N}^{2|4}) \;=\; \dim_{\mathbb{C}} H^0(\mathbb{C}P^1_{x,\eta},\mathcal{N}^{2|4}) \;=\; 4|8\ . \tag{2.35}$$

Furthermore, there are no obstructions to the deformation of the $\mathbb{C}P^{1|0}_{x,\eta}$ inside $\mathcal{P}^{3|4}$ since $h^1(\mathbb{C}P^1_{x,\eta},\mathcal{N}^{2|4}) = 0|0$.

On the other hand, one can fix only the even moduli $x^{\alpha\dot{\alpha}}$ and consider a holomorphic embedding $\mathbb{C}P^{1|4}_x\hookrightarrow\mathcal{P}^{3|4}$ defined by the equations

$$z^\alpha_\pm \;=\; x^{\alpha\dot{\alpha}}\lambda^\pm_{\dot{\alpha}}\ . \tag{2.36}$$

Recall that the normal bundle to $\mathbb{C}P^{1|0}_x\hookrightarrow\mathcal{P}^{3|0}$ is the rank two vector bundle $\mathcal{O}(1)\oplus\mathcal{O}(1)$. In the supercase, the formal definition of the normal bundle by the short exact sequence

$$0\;\to\; T\mathbb{C}P^{1|4}\;\to\; T\mathcal{P}^{3|4}|_{\mathbb{C}P^{1|4}}\;\to\;\mathcal{N}^{2|0}\;\to\;0 \tag{2.37}$$

yields that $\mathcal{N}^{2|0} = T\mathcal{P}^{3|4}|_{\mathbb{C}P^{1|4}}/T\mathbb{C}P^{1|4}$ is a rank two holomorphic vector bundle over $\mathbb{C}P^{1|4}$ which is (in the real case) locally spanned by the vector fields $\gamma_\pm V^\pm_\alpha$, where $V^\pm_\alpha$ is the complex conjugate of $\bar{V}^\pm_\alpha$. A global section of $\mathcal{N}^{2|0}$ over $\mathcal{U}_\pm\cap\mathbb{C}P^{1|4}$ is of the form $s_\pm = T^\alpha_\pm\gamma_\pm V^\pm_\alpha$. Obviously, the transformation of the components $T^\alpha_\pm$ from patch to patch is given by $T^\alpha_+ = \lambda_+ T^\alpha_-$, i.e. $\mathcal{N}^{2|0} = \mathcal{O}(1)\oplus\mathcal{O}(1)$.

2.6 Holomorphic Chern-Simons theory

Consider a trivial rank n complex vector bundle $\mathcal{E}$ over $\mathcal{P}^{3|4}_\varepsilon$ and a connection one-form $\mathcal{A}$ on $\mathcal{E}$. We define hCS theory by the action

$$S_{\mathrm{hCS}} \;:=\; \int_{\mathcal{Z}_\varepsilon}\Omega\wedge\mathrm{tr}\left(\mathcal{A}^{0,1}\wedge\bar{\partial}\mathcal{A}^{0,1}+\frac{2}{3}\mathcal{A}^{0,1}\wedge\mathcal{A}^{0,1}\wedge\mathcal{A}^{0,1}\right)\ , \tag{2.38}$$

where $\mathcal{A}^{0,1}$ is the $(0,1)$-part of $\mathcal{A}$ which we assume to satisfy the conditions $\bar{V}^i_\pm\lrcorner\mathcal{A}^{0,1} = 0$ and $\bar{V}^i_\pm(\bar{V}^\pm_a\lrcorner\mathcal{A}^{0,1}) = 0$ for $a = 1,2,3$. Furthermore, Ω is the holomorphic volume form (2.32) and $\mathcal{Z}_\varepsilon$ is the subspace of $\mathcal{P}^{3|4}_\varepsilon$ for which[3] $\bar{\eta}^\pm_i = 0$ [1]. The trace is taken over the gauge group $\mathrm{GL}(n,\mathbb{C})$.

The equations of motion for (2.38) read

$$\bar{\partial}\mathcal{A}^{0,1}+\mathcal{A}^{0,1}\wedge\mathcal{A}^{0,1}\;=\;0\ . \tag{2.39}$$

In the following, we will usually discuss them using the components

$$\mathcal{A}^\pm_\alpha\;:=\;\bar{V}^\pm_\alpha\lrcorner\mathcal{A}^{0,1}\ ,\quad \mathcal{A}_{\bar{\lambda}_\pm}\;:=\;\frac{\partial}{\partial\bar{\lambda}_\pm}\lrcorner\mathcal{A}^{0,1}\ ,\quad \mathcal{A}^i_\pm\;:=\;\bar{V}^i_\pm\lrcorner\mathcal{A}^{0,1}\ . \tag{2.40}$$

[3] This condition is not a contradiction to $\eta^\pm_i\neq 0$, but merely a restriction of all functions on $\mathcal{P}^{3|4}_\varepsilon$ to be holomorphic in the $\eta^\pm_i$.

in which (2.39) takes, e.g. on $\mathcal{U}_+$, the form

$$\bar{V}^+_\alpha \mathcal{A}^+_\beta - \bar{V}^+_\beta \mathcal{A}^+_\alpha + [\mathcal{A}^+_\alpha, \mathcal{A}^+_\beta] = 0 \ , \tag{2.41a}$$

$$\partial_{\bar{\lambda}_+} \mathcal{A}^+_\alpha - \bar{V}^+_\alpha \mathcal{A}_{\bar{\lambda}_+} + [\mathcal{A}_{\bar{\lambda}_+}, \mathcal{A}^+_\alpha] = 0 \ . \tag{2.41b}$$

Using these components, we can rewrite the action (2.38) as

$$S_{\mathrm{hCS}} \ := \ \int_{\mathcal{Z}_\varepsilon} \mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda} \wedge \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge E^1 \wedge E^2 \ \Omega^\eta \, \mathrm{tr}\, \varepsilon^{abc} \left(\mathcal{A}_a \bar{V}_b \mathcal{A}_c + \frac{2}{3} \mathcal{A}_a \mathcal{A}_b \mathcal{A}_c \right) \ . \tag{2.42}$$

Recall that we assumed in (2.38) that $\mathcal{A}^i_\pm = 0$. Moreover, note that in the twistor approach, one considers those gauge potentials $\mathcal{A}^{0,1}$ for which the components $\mathcal{A}_{\bar{\lambda}_\pm}$ can be gauged away [18] and thus restricts the solution space to (2.39) to a subset[4]. In this case, one can choose a gauge such that the superfield expansion of $\mathcal{A}^\pm_\alpha$ and $\mathcal{A}_{\bar{\lambda}_\pm}$ in $\eta^\pm_i$ and $\lambda_\pm$ are given by the formulæ [18]

$$\begin{aligned} \mathcal{A}^+_\alpha \ =& \ \lambda^{\dot\alpha}_+ A_{\alpha\dot\alpha}(x) + \eta^+_i \chi^i_\alpha(x) + \gamma_+ \frac{1}{2!} \eta^+_i \eta^+_j \, \hat{\lambda}^{\dot\alpha}_+ \, \phi^{ij}_{\alpha\dot\alpha}(x) + \\ & + \gamma^2_+ \frac{1}{3!} \eta^+_i \eta^+_j \eta^+_k \, \hat{\lambda}^{\dot\alpha}_+ \, \hat{\lambda}^{\dot\beta}_+ \, \tilde{\chi}^{ijk}_{\alpha\dot\alpha\dot\beta}(x) + \gamma^3_+ \frac{1}{4!} \eta^+_i \eta^+_j \eta^+_k \eta^+_l \, \hat{\lambda}^{\dot\alpha}_+ \, \hat{\lambda}^{\dot\beta}_+ \, \hat{\lambda}^{\dot\gamma}_+ \, G^{ijkl}_{\alpha\dot\alpha\dot\beta\dot\gamma}(x) \ , \end{aligned} \tag{2.43a}$$

$$\begin{aligned} \mathcal{A}_{\bar{\lambda}_+} \ =& \ \gamma^2_+ \eta^+_i \eta^+_j \, \phi^{ij}(x) - \gamma^3_+ \eta^+_i \eta^+_j \eta^+_k \, \hat{\lambda}^{\dot\alpha}_+ \, \tilde{\chi}^{ijk}_{\dot\alpha}(x) + \\ & + 2\gamma^4_+ \eta^+_i \eta^+_j \eta^+_k \eta^+_l \, \hat{\lambda}^{\dot\alpha}_+ \, \hat{\lambda}^{\dot\beta}_+ G^{ijkl}_{\dot\alpha\dot\beta}(x) \ . \end{aligned} \tag{2.43b}$$

Together with these expansions, the field equations (2.41) of hCS theory on $\mathcal{Z}_\varepsilon$ are reduced to the equations of motion of $\mathcal{N} = 4$ supersymmetric self-dual Yang-Mills (SDYM) theory on $\mathbb{R}^4_\varepsilon = (\mathbb{R}^4, g_\varepsilon)$ where $g_{-1} = \mathrm{diag}(+1, +1, +1, +1)$ and $g_{+1} = \mathrm{diag}(-1, -1, +1, +1)$ [1, 18].

Besides the gauge chosen above, there is also a gauge in which $\mathcal{A}_{\bar{\lambda}_\pm} = 0$ and $\mathcal{A}^i_\pm \neq 0$ [18] and after performing the (super-)gauge transformation

$$(\mathcal{A}^\pm_\alpha \neq 0, \mathcal{A}_{\bar{\lambda}_\pm} \neq 0, \mathcal{A}^i_\pm = 0) \ \xrightarrow{\ \varphi\ } \ (\tilde{\mathcal{A}}^\pm_\alpha \neq 0, \tilde{\mathcal{A}}_{\bar{\lambda}_\pm} \ = \ 0, \tilde{\mathcal{A}}^i_\pm \neq 0) \ , \tag{2.44}$$

the hCS field equations (2.41) are transformed to

$$\bar{V}^\pm_\alpha \tilde{\mathcal{A}}^\pm_\beta - \bar{V}^\pm_\beta \tilde{\mathcal{A}}^\pm_\alpha + [\tilde{\mathcal{A}}^\pm_\alpha, \tilde{\mathcal{A}}^\pm_\beta] \ = \ 0 \ , \tag{2.45a}$$

$$\bar{V}^i_\pm \tilde{\mathcal{A}}^\pm_\alpha + \bar{V}^\pm_\alpha \tilde{\mathcal{A}}^i_\pm + \{\tilde{\mathcal{A}}^i_\pm, \tilde{\mathcal{A}}^\pm_\alpha\} \ = \ 0 \ , \tag{2.45b}$$

$$\bar{V}^i_\pm \tilde{\mathcal{A}}^j_\pm + \bar{V}^j_\pm \tilde{\mathcal{A}}^i_\pm + \{\tilde{\mathcal{A}}^i_\pm, \tilde{\mathcal{A}}^j_\pm\} \ = \ 0 \ , \tag{2.45c}$$

$$\partial_{\bar{\lambda}_\pm} \tilde{\mathcal{A}}^\pm_\alpha \ = \ 0 \ , \tag{2.45d}$$

$$\partial_{\bar{\lambda}_\pm} \tilde{\mathcal{A}}^i_\pm \ = \ 0 \ . \tag{2.45e}$$

Note, however, that these equations cannot be obtained from an action principle. From (2.45d) and a generalized Liouville theorem, it follows that the non-trivial components of the gauge potential are linear in $\lambda_\pm$ and one can therefore write

$$\tilde{\mathcal{A}}^+_\alpha \ = \ \lambda^{\dot\alpha}_\pm \tilde{\mathcal{A}}_{\alpha\dot\alpha} \quad \text{and} \quad \tilde{\mathcal{A}}^i_\pm \ = \ \lambda^{\dot\alpha}_\pm \tilde{\mathcal{A}}^i_{\dot\alpha} \ . \tag{2.46}$$

[4]This subset contains in particular the vacuum solution $\mathcal{A}^{0,1} = 0$ and its vicinity.

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Substituting (2.46) into (2.45), we arrive at the equivalent equations

$$[\nabla_{\alpha\dot\alpha},\nabla_{\beta\dot\beta}] \;=\; \varepsilon_{\dot\alpha\dot\beta}\mathcal{F}_{\alpha\beta}, \quad [\nabla^i_{\dot\alpha},\nabla_{\beta\dot\beta}] \;=\; \varepsilon_{\dot\alpha\dot\beta}\mathcal{F}^i_\beta, \quad \{\nabla^i_{\dot\alpha},\nabla^j_{\dot\beta}\} \;=\; \varepsilon_{\dot\alpha\dot\beta}\mathcal{F}^{ij}\;, \tag{2.47}$$

where $\mathcal{F}_{\alpha\beta}$ and $\mathcal{F}^{ij}$ are symmetric and antisymmetric, respectively, in their two indices. Here, we have introduced the differential operators

$$\nabla_{\alpha\dot\alpha} \;:=\; \partial_{\alpha\dot\alpha}+\tilde{\mathcal{A}}_{\alpha\dot\alpha} \quad\text{and}\quad \nabla^i_{\dot\alpha} \;:=\; \partial^i_{\dot\alpha}+\tilde{\mathcal{A}}^i_{\dot\alpha}\;, \tag{2.48}$$

together with the self-dual super field strength with components $(\mathcal{F}_{\alpha\beta},\mathcal{F}^i_\beta,\mathcal{F}^{ij})$ for the super gauge potential $(\tilde{\mathcal{A}}_{\alpha\dot\alpha},\tilde{\mathcal{A}}^i_{\dot\alpha})$. The equations (2.47) are equivalent to (2.41), and the expansions in the fermionic coordinates $\eta^{\dot\alpha}_i$ of the components of the super gauge potential and the super field strength contain all the physical fields.

For convenience, we will impose additionally a transverse gauge condition [19], and demand that

$$\eta^{\dot\alpha}_i\tilde{\mathcal{A}}^i_{\dot\alpha} \;=\; 0\;. \tag{2.49}$$

The residual gauge symmetry is the ordinary gauge symmetry of supersymmetric SDYM theory. In this transverse gauge, the expansions of the gauge potential and the corresponding field strength in the fermionic coordinates are well known, see e.g. [20]. It is usually sufficient to know that

$$\begin{aligned}\tilde{\mathcal{A}}^\pm_\alpha \;&=\; \lambda^{\dot\alpha}_\pm\tilde{\mathcal{A}}_{\alpha\dot\alpha} \;=\; \lambda^{\dot\alpha}_\pm\left(A_{\alpha\dot\alpha}-\varepsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta}_i\chi^i_\alpha+\ldots-\frac{1}{12}\varepsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta}_i\eta^{\dot\gamma}_j\eta^{\dot\delta}_k\eta^{\dot\varepsilon}_l\nabla_{\alpha\dot\gamma}G^{ijkl}_{\dot\delta\dot\varepsilon}\right)\;, \\ \tilde{\mathcal{A}}^i_\pm \;&=\; \lambda^{\dot\alpha}_\pm\tilde{\mathcal{A}}^i_{\dot\alpha} \;=\; \lambda^{\dot\alpha}_\pm\left(\varepsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta}_j\phi^{ij}+\frac{2}{3}\varepsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta}_j\eta^{\dot\gamma}_k\tilde\chi^{ijk}_{\dot\gamma}+\frac{1}{4}\varepsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta}_j\eta^{\dot\gamma}_k\eta^{\dot\delta}_l\left(G^{ijkl}_{\dot\gamma\dot\delta}+\varepsilon_{\dot\gamma\dot\delta}\cdots\right)\right)\;,\end{aligned} \tag{2.50}$$

as this already determines the other terms in the expansion completely. The equations (2.47) are satisfied if and only if the $\mathcal{N}=4$ supersymmetric SDYM equations on $(\mathbb{R}^4,g_\varepsilon)$ hold for the physical fields appearing in the expansion (2.50).

2.7 Supersymmetric self-dual Yang-Mills theory

The field content of $\mathcal{N}=4$ supersymmetric SDYM theory is given by the supermultiplet $(f_{\alpha\beta},\chi^i_\alpha,\phi^{ij},\tilde\chi^{ijk}_{\dot\alpha},G^{ijkl}_{\dot\alpha\dot\beta})$, whose components have helicities $(+1,+\frac{1}{2},0,-\frac{1}{2},-1)$, respectively. They are combined into the action [21]

$$S_{\rm SDYM} \;=\; \int \mathrm{d}^4x\;\mathrm{tr}\;\left(G^{\dot\alpha\dot\beta}f_{\dot\alpha\dot\beta}+\frac{\varepsilon}{2}\varepsilon_{ijkl}\tilde\chi^{\dot\alpha ijk}\nabla_{\alpha\dot\alpha}\chi^{\alpha l}+\frac{\varepsilon}{2}\varepsilon_{ijkl}\phi^{ij}\Box\phi^{kl}+\varepsilon_{ijkl}\phi^{ij}\chi^{\alpha k}\chi^l_\alpha\right), \tag{2.51}$$

where we introduced the shorthand notations $\Box := \frac{1}{2}\nabla_{\alpha\dot\alpha}\nabla^{\alpha\dot\alpha}$ and $G^{\dot\alpha\dot\beta} := \frac{1}{2}\varepsilon_{ijkl}G^{\dot\alpha\dot\beta ijkl}$. Note that the trace now is taken over the Lie algebra $\mathrm{u}(n)$. This action can also be obtained by substituting (2.43) into (2.38) and integrating over $\eta^\pm_i$ and $\lambda_\pm$, $\bar\lambda_\pm$. The corresponding equations of motion[5] read

$$
\begin{aligned}
f_{\dot\alpha\dot\beta} &= 0 \, , \\
\nabla_{\alpha\dot\alpha}\chi^{\alpha i} &= 0 \, , \\
\Box\phi^{ij} &= -\frac{\varepsilon}{2}\{\chi^{\alpha i}, \chi^{j}_{\alpha}\} \, , \\
\nabla_{\alpha\dot\alpha}\tilde\chi^{\dot\alpha ijk} &= +2\varepsilon\,[\phi^{[ij}, \chi^{k]}_{\alpha}] \, , \\
\varepsilon^{\dot\alpha\dot\gamma}\nabla_{\alpha\dot\alpha}G^{ijkl}_{\dot\gamma\dot\delta} &= +\varepsilon\{\chi^{[i}_{\alpha}, \tilde\chi^{jkl]}_{\dot\delta}\} - \varepsilon\,[\phi^{[ij}, \nabla_{\alpha\dot\delta}\phi^{kl]}] \, ,
\end{aligned}
\tag{2.52}
$$

where we used the following decomposition of the field strength in self-dual and anti-self-dual parts:

$$
F_{\alpha\dot\alpha\beta\dot\beta} \;:=\; [\nabla_{\alpha\dot\alpha}, \nabla_{\beta\dot\beta}] \;=\; \varepsilon_{\dot\alpha\dot\beta} f_{\alpha\beta} + \varepsilon_{\alpha\beta} f_{\dot\alpha\dot\beta} \, . \tag{2.53}
$$

The supersymmetric SDYM equations for $\mathcal{N} < 4$ are obtained by considering the first $\mathcal{N}+1$ equations of (2.52). In addition, one has to restrict the R-symmetry indices $i, j, \ldots$ of all the fields to run from 1 to $\mathcal{N}$. One should stress that for $\mathcal{N} < 4$, the supersymmetric SDYM equations describe indeed a subsector of the corresponding full SYM theory. For $\mathcal{N} = 4$ the field contents of both the self-dual and the full theory are identical, but the interactions differ.

3. Matrix models

In this section, we construct four different matrix models. We start with dimensionally reducing $\mathcal{N} = 4$ SDYM theory to a point, which yields the first matrix model. The matrices here are just finite-dimensional matrices from the Lie algebra of the gauge group $\mathrm{U}(n)$. The second matrix model we consider results from a dimensional reduction of hCS theory on $\mathcal{P}^{3|4}_{\varepsilon}$ to a subspace $\mathcal{P}^{1|4}_{\varepsilon} \subset \mathcal{P}^{3|4}_{\varepsilon}$. We obtain a form of matrix quantum mechanics with a complex "time". This matrix model is linked by a Penrose-Ward transform to the first matrix model.

By considering again $\mathcal{N} = 4$ SDYM theory, but on noncommutative spacetime, we obtain a third matrix model. Here, we have finite-dimensional matrices with operator entries which can be realized as infinite-dimensional matrices acting on the tensor product of the gauge algebra representation space and the Fock space. The fourth and last matrix model is obtained by rendering the fibre coordinates in the vector bundle $\mathcal{P}^{3|4}_{\varepsilon} \to \mathbb{C}P^{1|4}$ noncommutative. In the operator formulation, this again yields a matrix model with infinite-dimensional matrices and there is also a Penrose-Ward transform which renders the two noncommutative matrix models equivalent.

In a certain limit, in which the rank n of the gauge groups $\mathrm{U}(n)$ and $\mathrm{GL}(n, \mathbb{C})$ of the SDYM and the hCS matrix model tends to infinity, one expects them to become equivalent to the respective matrix models obtained from noncommutativity.

[5] Recall that these equations are equivalent to both the holomorphic Chern-Simons equations (2.41) with the expansion (2.43) and the equations (2.45) with the expansion (2.50).

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3.1 Matrix model of $\mathcal{N}=4$ SDYM theory

We start from the Lagrangian in the action (2.51) of $\mathcal{N}=4$ supersymmetric self-dual Yang-Mills theory in four dimensions with gauge group $\mathrm{U}(n)$. One can dimensionally reduce this theory to a point by assuming that all the fields are independent of $x\in\mathbb{R}^4$. This yields the matrix model action

$$
\begin{aligned}
S_{\mathrm{SDYMMM}} \;=\; \operatorname{tr}\Bigg(& G^{\dot\alpha\dot\beta}\left(-\frac{1}{2}\varepsilon^{\alpha\beta}[A_{\alpha\dot\alpha},A_{\beta\dot\beta}]\right)+\frac{\varepsilon}{2}\varepsilon_{ijkl}\tilde\chi^{\dot\alpha ijk}[A_{\alpha\dot\alpha},\chi^{\alpha l}] \\
& +\frac{\varepsilon}{4}\varepsilon_{ijkl}\phi^{ij}[A_{\alpha\dot\alpha},[A^{\alpha\dot\alpha},\phi^{kl}]]+\varepsilon_{ijkl}\phi^{ij}\chi^{\alpha k}\chi^{l}_{\alpha}\Bigg)\,,
\end{aligned}
\tag{3.1}
$$

which is invariant under the adjoint action of the gauge group $\mathrm{U}(n)$ on all the fields. This symmetry is the remnant of gauge invariance. The corresponding equations of motion read

$$
\begin{aligned}
\varepsilon^{\alpha\beta}[A_{\alpha\dot\alpha},A_{\beta\dot\beta}] &= 0\,,\\
[A_{\alpha\dot\alpha},\chi^{\alpha i}] &= 0\,,\\
\frac{1}{2}[A^{\alpha\dot\alpha},[A_{\alpha\dot\alpha},\phi^{ij}]] &= -\frac{\varepsilon}{2}\{\chi^{\alpha i},\chi^{j}_{\alpha}\}\,,\\
[A_{\alpha\dot\alpha},\tilde\chi^{\dot\alpha ijk}] &= +2\varepsilon\,[\phi^{ij},\chi^{k}_{\alpha},]\,,\\
\varepsilon^{\dot\alpha\dot\gamma}[A_{\alpha\dot\alpha},G^{ijkl}_{\dot\gamma\dot\delta}] &= +\varepsilon\{\chi^{i}_{\alpha},\tilde\chi^{jkl}_{\dot\delta}\}-\varepsilon\,[\phi^{ij},[A_{\alpha\dot\delta},\phi^{kl}]]\,.
\end{aligned}
\tag{3.2}
$$

Note that these equations can be obtained by dimensionally reducing equations (2.52) to a point. On the other hand, the equations of motion of $\mathcal{N}=4$ SDYM theory are equivalent to the constraint equations (2.47) which are defined on the superspace $\mathbb{R}^{4|8}$. Therefore, (3.2) are equivalent to the equations

$$
[\tilde{\mathcal{A}}_{\alpha\dot\alpha},\tilde{\mathcal{A}}_{\beta\dot\beta}] \;=\; \varepsilon_{\dot\alpha\dot\beta}\mathcal{F}_{\alpha\beta}\,,\qquad \nabla^{i}_{\dot\alpha}\tilde{\mathcal{A}}_{\beta\dot\beta} \;=\; \varepsilon_{\dot\alpha\dot\beta}\mathcal{F}^{i}_{\beta}\,,\qquad \{\nabla^{i}_{\dot\alpha},\nabla^{j}_{\dot\beta}\} \;=\; \varepsilon_{\dot\alpha\dot\beta}\mathcal{F}^{ij} \tag{3.3}
$$

obtained from (2.47) by dimensional reduction to the supermanifold[6] $\mathbb{R}^{0|8}$.

Recall that the IKKT matrix model [22] can be obtained by dimensionally reducing $\mathcal{N}=1$ SYM theory in ten dimensions or $\mathcal{N}=4$ SYM in four dimensions to a point. In this sense, the above matrix model is the self-dual analogue of the IKKT matrix model.

3.2 Matrix model from hCS theory

So far, we have constructed a matrix model for $\mathcal{N}=4$ SDYM theory, the latter being defined on the space $(\mathbb{R}^{4|8},g_\varepsilon)$ with $\varepsilon=-1$ corresponding to Euclidean signature and $\varepsilon=+1$ corresponding to Kleinian signature of the metric on $\mathbb{R}^4$. The next step is evidently to ask what theory corresponds to the matrix model introduced above on the twistor space side.

Recall that for the two signatures on $\mathbb{R}^4$, we use the supertwistor spaces

$$
\mathcal{P}^{3|4}_{\varepsilon} \;\cong\; \Sigma^{1}_{\varepsilon}\times\mathbb{R}^{4|8} \tag{3.4}
$$

[6] Following the usual nomenclature of superlines and superplanes, this would be a "superpoint".

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where

$$\Sigma^1_{-1} \;:=\; \mathbb{C}P^1 \quad \text{and} \quad \Sigma^1_{+1} \;:=\; H^2 \tag{3.5}$$

and the two-sheeted hyperboloid H^2 is considered as a complex space. As was discussed in section 3.1, the equations of motion (3.2) of the matrix model (3.1) can be obtained from the constraint equations (2.47) by reducing the space $\mathbb{R}^{4|8}$ to the supermanifold $\mathbb{R}^{0|8}$ and expanding the superfields contained in (3.3) in the Graßmann variables $\eta_i^{\dot\alpha}$. On the twistor space side, this reduction yields the orbit spaces

$$\Sigma^1_\varepsilon \times \mathbb{R}^{0|8} \;=\; \mathcal{P}^{3|4}_\varepsilon/\mathscr{G} \;, \tag{3.6}$$

where $\mathscr{G}$ is the abelian group of translations generated by the bosonic vector fields $\frac{\partial}{\partial x^{\alpha\dot\alpha}}$. Equivalently, one can define the spaces $\mathcal{P}^{1|4}_\varepsilon$ as the orbit spaces

$$\mathcal{P}^{1|4}_\varepsilon \;:=\; \mathcal{P}^{3|4}_\varepsilon/\mathscr{G}^{1,0} \;, \tag{3.7}$$

where $\mathscr{G}^{1,0}$ is the complex abelian group generated by the vector fields $\frac{\partial}{\partial z^\alpha_\pm}$. These spaces with $\varepsilon = \pm 1$ are covered by the two patches $U^\varepsilon_\pm \cong \mathbb{C}^{1|4}$ and they are obviously diffeomorphic to the spaces (3.6), i.e.

$$\mathcal{P}^{1|4}_\varepsilon \;\cong\; \Sigma^1_\varepsilon \times \mathbb{R}^{0|8} \tag{3.8}$$

due to the diffeomorphism (3.4). In the coordinates $(z^3_\pm, \eta_i^\pm)$ on $\mathcal{P}^{1|4}_\varepsilon$ and $(\lambda_\pm, \eta_i^{\dot\alpha})$ on $\Sigma^1_\varepsilon \times \mathbb{R}^{0|8}$, the diffeomorphism is defined e.g. by the formulæ

$$\begin{aligned}
\eta_1^{\dot 1} &= \frac{\eta_1^+ - z^3_+\bar\eta_2^+}{1+z^3_+\bar z^3_+} = \frac{\bar z^3_-\eta_1^- - \bar\eta_2^-}{1+z^3_-\bar z^3_-}\;, \quad & \eta_2^{\dot 1} &= \frac{\eta_2^+ + z^3_+\bar\eta_1^+}{1+z^3_+\bar z^3_+} = \frac{\bar z^3_-\eta_2^- + \bar\eta_1^-}{1+z^3_-\bar z^3_-}\;,\\
\eta_3^{\dot 1} &= \frac{\eta_3^+ - z^3_+\bar\eta_4^+}{1+z^3_+\bar z^3_+} = \frac{\bar z^3_-\eta_3^- - \bar\eta_4^-}{1+z^3_-\bar z^3_-}\;, \quad & \eta_4^{\dot 1} &= \frac{\eta_4^+ + z^3_+\bar\eta_3^+}{1+z^3_+\bar z^3_+} = \frac{\bar z^3_-\eta_4^- + \bar\eta_3^-}{1+z^3_-\bar z^3_-}\;,
\end{aligned} \tag{3.9}$$

in the Euclidean case $\varepsilon = -1$. Thus, we have a dimensionally reduced twistor correspondence between the spaces $\mathcal{P}^{1|4}_\varepsilon$ and $\mathbb{R}^{0|8}$

$$\begin{tikzcd}
 & \mathbb{R}^{0|8}\times\Sigma^1_\varepsilon \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\
\mathcal{P}^{1|4}_\varepsilon & & \mathbb{R}^{0|8}
\end{tikzcd} \tag{3.10}$$

where the map π_2 is the diffeomorphism (3.8). It follows from (3.10) that we have a correspondence between points $\eta \in \mathbb{R}^{0|8}$ and subspaces $\mathbb{C}P^1_\eta$ of $\mathcal{P}^{1|4}_\varepsilon$.

Holomorphic Chern-Simons theory on $\mathcal{P}^{3|4}_\varepsilon$ with the action (2.38) is defined by a gauge potential $\mathcal{A}^{0,1}$ taking values in the Lie algebra of $\mathrm{GL}(n,\mathbb{C})$ and constrained by the equations $\bar V^i_\pm(\bar V^\pm_a \lrcorner \mathcal{A}^{0,1}) = 0$, $\bar V^i_\pm \lrcorner \mathcal{A}^{0,1} = 0$ for $a = 1,2,3$. After reduction to $\mathcal{P}^{1|4}_\varepsilon$, $\mathcal{A}^{0,1}$ splits into a gauge potential and two complex scalar fields taking values in the normal bundle $\mathbb{C}^2 \otimes \mathcal{O}(1)$ to the space $\mathcal{P}^{1|4}_\varepsilon \hookrightarrow \mathcal{P}^{3|4}_\varepsilon$. In components, we have

$$\mathcal{A}^{0,1}_{\Sigma\pm} \;=\; \mathrm{d}\bar\lambda_\pm \mathcal{A}_{\bar\lambda_\pm} \quad \text{and} \quad \mathcal{A}^\pm_\alpha \;\to\; \mathcal{X}^\pm_\alpha \quad \text{on} \quad U^\varepsilon_\pm \;, \tag{3.11}$$

where both $\mathcal{X}^\pm_\alpha$ and $\mathcal{A}_{\bar\lambda_\pm}$ are Lie algebra valued superfunctions on the subspaces $U^\varepsilon_\pm$ of $\mathcal{P}^{1|4}_\varepsilon$. The integral over the chiral subspace $\mathcal{Z}_\varepsilon \subset \mathcal{P}^{3|4}_\varepsilon$ should be evidently substituted by an integral over the chiral subspace $\mathcal{Y}_\varepsilon \subset \mathcal{P}^{1|4}_\varepsilon$. This dimensional reduction of the bosonic

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coordinates becomes even clearer with the help of the identity

$$\mathrm{d}\lambda_\pm \wedge \mathrm{d}\bar{\lambda}_\pm \wedge \mathrm{d}z^1_\pm \wedge \mathrm{d}z^2_\pm \wedge \bar{E}^1_\pm \wedge \bar{E}^2_\pm \;=\; \mathrm{d}\lambda_\pm \wedge \mathrm{d}\bar{\lambda}_\pm \wedge \mathrm{d}x^{1\dot{1}} \wedge \mathrm{d}x^{1\dot{2}} \wedge \mathrm{d}x^{2\dot{1}} \wedge \mathrm{d}x^{2\dot{2}}\ . \tag{3.12}$$

Altogether, the dimensionally reduced action reads

$$S_{\mathrm{hCS,red}} \;:=\; \int_{\mathcal{Y}_\varepsilon} \omega \wedge \operatorname{tr} \varepsilon^{\alpha\beta} \mathcal{X}_\alpha \left(\bar{\partial} \mathcal{X}_\beta + \left[\mathcal{A}^{0,1}_\Sigma, \mathcal{X}_\beta \right] \right)\ , \tag{3.13}$$

where the form ω has components

$$\omega_\pm \;:=\; \Omega|_{U^\varepsilon_\pm} \;=\; \pm \mathrm{d}\lambda_\pm \mathrm{d}\eta^\pm_1 \ldots \mathrm{d}\eta^\pm_4 \tag{3.14}$$

and thus takes values in the bundle $\mathcal{O}(-2)$. Note furthermore that $\bar{\partial}$ here is the Dolbeault operator on Σ^1_ε and the integral in (3.13) is well-defined since the $\mathcal{X}_\alpha$ take values in the bundles $\mathcal{O}(1)$. The corresponding equations of motion are given by

$$[\mathcal{X}_1, \mathcal{X}_2] = 0\ , \tag{3.15a}$$

$$\bar{\partial}\mathcal{X}_\alpha + [\mathcal{A}^{0,1}_\Sigma, \mathcal{X}_\alpha] = 0\ . \tag{3.15b}$$

The gauge symmetry is obviously reduced to the transformations

$$\mathcal{X}_\alpha \;\to\; \varphi^{-1}\mathcal{X}_\alpha\varphi \quad \text{and} \quad \mathcal{A}^{0,1}_\Sigma \;\to\; \varphi^{-1}\mathcal{A}^{0,1}_\Sigma\varphi + \varphi^{-1}\bar{\partial}\varphi\ , \tag{3.16}$$

where φ is a smooth $\mathrm{GL}(n,\mathbb{C})$-valued function on $\mathcal{P}^{1|4}_\varepsilon$. The matrix model given by (3.13) and the field equations (3.15) can be understood as matrix quantum mechanics with a complex "time" $\lambda \in \Sigma^1_\varepsilon$.

Both the matrix models obtained by dimensional reductions of $\mathcal{N} = 4$ supersymmetric SDYM theory and hCS theory are (classically) equivalent. This follows from the dimensional reduction of the formulæ (2.43) defining the Penrose-Ward transform. The reduced superfield expansion is fixed by the geometry of $\mathcal{P}^{1|4}_\varepsilon$ and reads explicitly as

$$\begin{aligned} \mathcal{X}^+_\alpha \;=\;& \lambda^{\dot{\alpha}}_+ A_{\alpha\dot{\alpha}} + \eta^+_i \chi^i_\alpha + \gamma_+ \frac{1}{2!}\eta^+_i\eta^+_j\, \hat{\lambda}^{\dot{\alpha}}_+\, \phi^{ij}_{\alpha\dot{\alpha}} + \\ &+ \gamma^2_+ \frac{1}{3!}\eta^+_i\eta^+_j\eta^+_k\, \hat{\lambda}^{\dot{\alpha}}_+\, \hat{\lambda}^{\dot{\beta}}_+\, \tilde{\chi}^{ijk}_{\alpha\dot{\alpha}\dot{\beta}} + \gamma^3_+ \frac{1}{4!}\eta^+_i\eta^+_j\eta^+_k\eta^+_l\, \hat{\lambda}^{\dot{\alpha}}_+\, \hat{\lambda}^{\dot{\beta}}_+\, \hat{\lambda}^{\dot{\gamma}}_+\, G^{ijkl}_{\alpha\dot{\alpha}\dot{\beta}\dot{\gamma}}\ , \end{aligned} \tag{3.17a}$$

$$\begin{aligned} \mathcal{A}_{\bar{\lambda}_+} \;=\;& \gamma^2_+\eta^+_i\eta^+_j\, \phi^{ij} - \gamma^3_+\eta^+_i\eta^+_j\eta^+_k\, \hat{\lambda}^{\dot{\alpha}}_+\, \tilde{\chi}^{ijk}_{\dot{\alpha}} + \\ &+ 2\gamma^4_+\eta^+_i\eta^+_j\eta^+_k\eta^+_l\, \hat{\lambda}^{\dot{\alpha}}_+\, \hat{\lambda}^{\dot{\beta}}_+ G^{ijkl}_{\dot{\alpha}\dot{\beta}}\ , \end{aligned} \tag{3.17b}$$

where all component fields are independent of $x \in \mathbb{R}^4$. One can substitute this expansion into the action (3.13) and after a subsequent integration over $\mathcal{P}^{1|4}_\varepsilon$, one obtains the action (3.1) up to a constant multiplier, which is the volume[7] of Σ^1_ε.

[7]In the Kleinian case, this volume is naïvely infinite, but one can regularize it by utilizing a suitable partition of unity.

3.3 Noncommutative $\mathcal{N}=4$ SDYM theory

Noncommutative field theories have received much attention recently, as they were found to arise in string theory in the presence of D-branes and a constant NS B-field background [23–25].

There are two completely equivalent ways of introducing a noncommutative deformation of classical field theory: a star-product formulation and an operator formalism. In the first approach, one simply deforms the ordinary product of classical fields (or their components) to the noncommutative star product which reads in spinor notation as

$$(f \star g)(x) \;:=\; f(x) \exp\left(\frac{\mathrm{i}}{2}\overleftarrow{\partial_{\alpha\dot\alpha}}\theta^{\alpha\dot\alpha\beta\dot\beta}\overrightarrow{\partial_{\beta\dot\beta}}\right) g(x) \tag{3.18}$$

with $\theta^{\alpha\dot\alpha\beta\dot\beta} = -\theta^{\beta\dot\beta\alpha\dot\alpha}$ and in particular

$$x^{\alpha\dot\alpha} \star x^{\beta\dot\beta} - x^{\beta\dot\beta} \star x^{\alpha\dot\alpha} \;=\; \mathrm{i}\theta^{\alpha\dot\alpha\beta\dot\beta} \ . \tag{3.19}$$

In the following, we restrict ourselves to the case of a self-dual ($\kappa = 1$) or an anti-self-dual ($\kappa = -1$) tensor $\theta^{\alpha\dot\alpha\beta\dot\beta}$ and choose coordinates such that

$$\theta^{1\dot 1 2\dot 2} \;=\; -\theta^{2\dot 2 1\dot 1} \;=\; -2\mathrm{i}\kappa\varepsilon\theta \quad \text{and} \quad \theta^{1\dot 2 2\dot 1} \;=\; -\theta^{2\dot 1 1\dot 2} \;=\; 2\mathrm{i}\varepsilon\theta \ . \tag{3.20}$$

The formulation of noncommutative $\mathcal{N}=4$ SDYM theory on $(\mathbb{R}^4_\theta, g_\varepsilon)$ is now achieved by replacing all products in the action (2.51) by star products. For example, the noncommutative field strength will read

$$F_{\alpha\dot\alpha\beta\dot\beta} \;=\; \partial_{\alpha\dot\alpha}A_{\beta\dot\beta} - \partial_{\beta\dot\beta}A_{\alpha\dot\alpha} + A_{\alpha\dot\alpha} \star A_{\beta\dot\beta} - A_{\beta\dot\beta} \star A_{\alpha\dot\alpha} \ . \tag{3.21}$$

For the matrix reformulation of our model, it is necessary to switch to the operator formalism, which trades the star product for operator-valued coordinates $\hat{x}^{\alpha\dot\alpha}$ satisfying

$$[\hat{x}^{\alpha\dot\alpha}, \hat{x}^{\beta\dot\beta}] \;=\; \mathrm{i}\theta^{\alpha\dot\alpha\beta\dot\beta} \ . \tag{3.22}$$

This defines the noncommutative space $\mathbb{R}^4_\theta$ and on this space, derivatives are inner derivations of the Heisenberg algebra (3.22):

$$\begin{aligned} \frac{\partial}{\partial \hat{x}^{1\dot 1}} f \;&:=\; -\frac{1}{2\kappa\varepsilon\theta}[\hat{x}^{2\dot 2}, f] \ , \qquad & \frac{\partial}{\partial \hat{x}^{2\dot 2}} f \;&:=\; +\frac{1}{2\kappa\varepsilon\theta}[\hat{x}^{1\dot 1}, f] \ , \\ \frac{\partial}{\partial \hat{x}^{1\dot 2}} f \;&:=\; +\frac{1}{2\varepsilon\theta}[\hat{x}^{2\dot 1}, f] \ , \qquad & \frac{\partial}{\partial \hat{x}^{1\dot 2}} f \;&:=\; -\frac{1}{2\varepsilon\theta}[\hat{x}^{1\dot 2}, f] \ . \end{aligned} \tag{3.23}$$

The obvious representation space for the algebra (3.22) is the two-oscillator Fock space $\mathcal{H}$ which is created from a vacuum state $|0,0\rangle$. This vacuum state is annihilated by the operators

$$\hat{a}_1 \;=\; \mathrm{i}\left(\frac{1-\varepsilon}{2}\hat{x}^{2\dot 1} + \frac{1+\varepsilon}{2}\hat{x}^{1\dot 2}\right) \quad \text{and} \quad \hat{a}_2 \;=\; -\mathrm{i}\left(\frac{1-\kappa\varepsilon}{2}\hat{x}^{2\dot 2} + \frac{1+\kappa\varepsilon}{2}\hat{x}^{1\dot 1}\right) \tag{3.24}$$

and all other states of $\mathcal{H}$ are obtained by acting with the corresponding creation operators on $|0,0\rangle$. Thus, coordinates as well as fields are to be regarded as operators in $\mathcal{H}$.

Via the Moyal-Weyl map [23 – 25], any function $\Phi(x)$ in the star-product formulation can be related to an operator-valued function $\hat{\Phi}(\hat{x})$ acting in $\mathcal{H}$. This yields the operator equivalent of star multiplication and integration

$$f \star g \;\mapsto\; \hat{f}\hat{g} \quad \text{and} \quad \int \mathrm{d}^4 x f \;\mapsto\; (2\pi\theta)^2 \,\mathrm{tr}_{\mathcal{H}} \hat{f} \;, \tag{3.25}$$

where $\mathrm{tr}_{\mathcal{H}}$ signifies the trace over the Fock space $\mathcal{H}$.

We now have all the ingredients for defining noncommutative $\mathcal{N} = 4$ super SDYM theory in the operator formalism. Starting point is the analogue of the covariant derivatives which are given by the formulæ

$$\begin{aligned}
\hat{X}_{1\dot{1}} &= -\frac{1}{2\kappa\varepsilon\theta}\hat{x}^{2\dot{2}} \otimes \mathbb{1}_n + \hat{A}_{1\dot{1}} \;, \quad & \hat{X}_{2\dot{2}} &= \frac{1}{2\kappa\varepsilon\theta}\hat{x}^{1\dot{1}} \otimes \mathbb{1}_n + \hat{A}_{2\dot{2}} \;, \\
\hat{X}_{1\dot{2}} &= \frac{1}{2\varepsilon\theta}\hat{x}^{2\dot{1}} \otimes \mathbb{1}_n + \hat{A}_{1\dot{2}} \;, \quad & \hat{X}_{2\dot{1}} &= -\frac{1}{2\varepsilon\theta}\hat{x}^{1\dot{2}} \otimes \mathbb{1}_n + \hat{A}_{2\dot{1}} \;.
\end{aligned}$$

These operators act on the tensor product of the Fock space $\mathcal{H}$ and the representation space of the Lie algebra of the gauge group $\mathrm{U}(n)$. The operator-valued field strength has then the form

$$\hat{F}_{\alpha\dot{\alpha}\beta\dot{\beta}} \;=\; [\hat{X}_{\alpha\dot{\alpha}}, \hat{X}_{\beta\dot{\beta}}] + \mathrm{i}\theta_{\alpha\dot{\alpha}\beta\dot{\beta}} \otimes \mathbb{1}_n \;, \tag{3.26}$$

where the tensor $\theta_{\alpha\dot{\alpha}\beta\dot{\beta}}$ has components

$$\theta_{1\dot{1}2\dot{2}} \;=\; -\theta_{2\dot{2}1\dot{1}} \;=\; \mathrm{i}\frac{\kappa\varepsilon}{2\theta} \;, \quad \theta_{1\dot{2}2\dot{1}} \;=\; -\theta_{2\dot{1}1\dot{2}} \;=\; -\mathrm{i}\frac{\varepsilon}{2\theta} \;, \tag{3.27}$$

Recall that noncommutativity restricts the set of allowed gauge groups and we therefore had to choose to work with $\mathrm{U}(n)$ instead of $\mathrm{SU}(n)$.

The action of noncommutative SDYM theory on $(\mathbb{R}^4_\theta, g_\varepsilon)$ reads

$$\begin{aligned}
S^{\mathcal{N}=4}_{\mathrm{ncSDYM}} \;=\; \mathrm{tr}_{\mathcal{H}}\,\mathrm{tr}\Bigg(& -\frac{1}{2}\varepsilon^{\alpha\beta}\hat{G}^{\dot{\alpha}\dot{\beta}}\left([\hat{X}_{\alpha\dot{\alpha}}, \hat{X}_{\beta\dot{\beta}}] + \mathrm{i}\theta_{\alpha\dot{\alpha}\beta\dot{\beta}} \otimes \mathbb{1}_n\right) \\
& + \frac{\varepsilon}{2}\varepsilon_{ijkl}\tilde{\hat{\chi}}^{\dot{\alpha}ijk}[\hat{X}_{\alpha\dot{\alpha}}, \hat{\chi}^{\alpha l}] + \frac{\varepsilon}{2}\varepsilon_{ijkl}\hat{\phi}^{ij}[\hat{X}_{\alpha\dot{\alpha}}, [\hat{X}^{\alpha\dot{\alpha}}, \hat{\phi}^{kl}]] + \varepsilon_{ijkl}\hat{\phi}^{ij}\hat{\chi}^{\alpha k}\hat{\chi}^l_\alpha \Bigg) \;.
\end{aligned} \tag{3.28}$$

For $\kappa = +1$, the term containing $\theta_{\alpha\dot{\alpha}\beta\dot{\beta}}$ vanishes after performing the index sums. Note furthermore that in the limit of $n \to \infty$ for the gauge group $\mathrm{U}(n)$, one can render the ordinary $\mathcal{N} = 4$ SDYM matrix model (3.1) equivalent to noncommutative $\mathcal{N} = 4$ SDYM theory defined by the action (3.28). This is based on the fact that there is an isomorphism of spaces $\mathbb{C}^\infty \cong \mathcal{H}$ and $\mathbb{C}^n \otimes \mathcal{H}$.

3.4 Noncommutative hCS theory

The natural question to ask at this point is whether one can translate the Penrose-Ward transform completely into the noncommutative situation and therefore obtain a holomorphic Chern-Simons theory on a noncommutative supertwistor space. For the Penrose-Ward transform in the purely bosonic case, the answer is positive (see e.g. [26 – 28]).

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In the supersymmetric case, simply by taking the correspondence space to be the product space $(\mathbb{R}^{4|8}_\theta, g_\varepsilon)\times\Sigma^1_\varepsilon$ with the coordinate algebra (3.22) and unchanged algebra of Graßmann coordinates, we arrive together with the incidence relation in (2.7) at noncommutative coordinates[8] on the twistor space $\mathcal{P}^{3|4}_{\varepsilon,\theta}$ satisfying the relations

$$\begin{aligned}
[\hat z^1_\pm,\hat z^2_\pm] &= 2(\kappa-1)\varepsilon\lambda_\pm\theta\ , & [\hat{\bar z}^1_\pm,\hat{\bar z}^2_\pm] &= -2(\kappa-1)\varepsilon\bar\lambda_\pm\theta\ ,\\
[\hat z^1_+,\hat{\bar z}^1_+] &= 2(\kappa\varepsilon-\lambda_+\bar\lambda_+)\theta\ , & [\hat z^1_-,\hat{\bar z}^1_-] &= 2(\kappa\varepsilon\lambda_-\bar\lambda_- -1)\theta\ ,\\
[\hat z^2_+,\hat{\bar z}^2_+] &= 2(1-\varepsilon\kappa\lambda_+\bar\lambda_+)\theta\ , & [\hat z^2_-,\hat{\bar z}^2_-] &= 2(\lambda_-\bar\lambda_- -\varepsilon\kappa)\theta\ ,
\end{aligned} \tag{3.29}$$

with all other commutators vanishing. Here, we clearly see the advantage of choosing a self-dual deformation tensor $\kappa=+1$: the first line in (3.29) becomes trivial. We will restrict our considerations to this case[9] in the following.

Thus, we see that the coordinates z^α and $\bar z^\alpha$ are turned into sections $\hat z^\alpha$ and $\hat{\bar z}^\alpha$ of the bundle $\mathcal{O}(1)$ which are functions on $\mathcal{P}^{1|4}_\varepsilon$ and take values in the the space of operators acting on the Fock space $\mathcal{H}$. The derivatives along the bosonic fibres of the fibration $\mathcal{P}^{3|4}_\varepsilon\to\mathcal{P}^{1|4}_\varepsilon$ are turned into inner derivatives of the algebra (3.29):

$$\frac{\partial}{\partial\hat{\bar z}^1_\pm}f \;=\; \frac{\varepsilon}{2\theta}\gamma_\pm\,[\hat z^1_\pm,f]\ ,\quad \frac{\partial}{\partial\hat{\bar z}^2_\pm}f \;=\; \frac{1}{2\theta}\gamma_\pm\,[\hat z^2_\pm,f]\ . \tag{3.30}$$

Together with the identities (2.27), we can furthermore derive

$$\hat{\bar V}{}^\pm_1 f \;=\; -\frac{\varepsilon}{2\theta}\,[\hat z^2_\pm,f] \quad\text{and}\quad \hat{\bar V}{}^\pm_2 f \;=\; -\frac{\varepsilon}{2\theta}\,[\hat z^1_\pm,f]\ . \tag{3.31}$$

The formulæ (3.31) allow us to define the noncommutatively deformed version of the hCS action (2.42):

$$\begin{aligned}
S_{\mathrm{nchCS}} \;:=\; \int_{\mathcal{Y}_\varepsilon}\omega\wedge\mathrm{tr}_{\mathcal{H}}\,\mathrm{tr}\,\Big\{&\Big(\hat{\mathcal{A}}_2\bar\partial\hat{\mathcal{A}}_1-\hat{\mathcal{A}}_1\bar\partial\hat{\mathcal{A}}_2\Big)+2\hat{\mathcal{A}}^{0,1}_\Sigma\Big[\hat{\mathcal{A}}_1,\hat{\mathcal{A}}_2\Big]- \\
&-\frac{\varepsilon}{2\theta}\Big(\hat{\mathcal{A}}_1\Big[\hat z^1,\hat{\mathcal{A}}^{0,1}_\Sigma\Big]-\hat{\mathcal{A}}^{0,1}_\Sigma\Big[\hat z^1,\hat{\mathcal{A}}_1\Big]+\hat{\mathcal{A}}^{0,1}_\Sigma\Big[\hat z^2,\hat{\mathcal{A}}_2\Big]-\hat{\mathcal{A}}_2\Big[\hat z^2,\hat{\mathcal{A}}^{0,1}_\Sigma\Big]\Big)\Big\}\ ,
\end{aligned} \tag{3.32}$$

where $\mathcal{Y}_\varepsilon$ is again the chiral subspace of $\mathcal{P}^{1|4}_\varepsilon$ for which $\bar\eta^i_\pm=0$, ω is the form defined in (3.14) and $\mathrm{tr}_{\mathcal{H}}$ and tr denote the traces over the Fock space $\mathcal{H}$ and the representation space of $\mathrm{gl}(n,\mathbb{C})$, respectively. The hats indicate that the components of the gauge potential $\hat{\mathcal{A}}^{0,1}$ are now operators with values in the Lie algebra $\mathrm{gl}(n,\mathbb{C})$.

We can simplify the above action by introducing the operators

$$\hat{\mathcal{X}}^1_\pm \;=\; -\frac{\varepsilon}{2\theta}\hat z^2_\pm\otimes\mathbb{1}_n+\hat{\mathcal{A}}^\pm_1 \quad\text{and}\quad \hat{\mathcal{X}}^2_\pm \;=\; -\frac{\varepsilon}{2\theta}\hat z^1_\pm\otimes\mathbb{1}_n+\hat{\mathcal{A}}^\pm_2 \tag{3.33}$$

which yields

$$S_{\mathrm{nchCS}} \;=\; \int_{\mathcal{Y}_\varepsilon}\omega\wedge\mathrm{tr}_{\mathcal{H}}\,\mathrm{tr}\,\varepsilon^{\alpha\beta}\hat{\mathcal{X}}_\alpha\Big(\bar\partial\hat{\mathcal{X}}_\beta+[\hat{\mathcal{A}}^{0,1}_\Sigma,\hat{\mathcal{X}}_\beta]\Big)\ , \tag{3.34}$$

where the $\hat{\mathcal{X}}_\alpha$ take values in the bundle $\mathcal{O}(1)$, so that the above integral is indeed well defined. Note that in the matrix model (3.13), we considered matrices taking values in the Lie algebra $\mathrm{gl}(n,\mathbb{C})$, while the fields $\hat{\mathcal{X}}_\alpha$ and $\hat{\mathcal{A}}^{0,1}_\Sigma$ in the model (3.34) take values in $\mathrm{gl}(n,\mathbb{C})\otimes\mathrm{End}\,(\mathcal{H})$ and can be represented by infinite dimensional matrices.

[8] Observe that the coordinates on Σ^1_ε stay commutative.

[9] Recall, however, that the singularities of the moduli space of self-dual solutions are not resolved when choosing a self-dual deformation tensor.

3.5 String field theory

It is of interest to generalize the twistor correspondence to the level of string field theory (SFT). This could be done using the approaches [29] or [30]. Alternatively, one could concentrate on (an appropriate extension of) SFT for $\mathcal{N}=2$ string theory [31]. At tree level, open $\mathcal{N}=2$ strings are known to reduce to the SDYM model in a Lorentz noninvariant gauge [32]; their SFT formulation [33] is based on the $\mathcal{N}=4$ topological string description [34]. The latter contains twistors from the outset: The coordinate $\lambda \in \mathbb{C}P^1$, the linear system, and the classical solutions with the help of twistor methods were all incorporated in $\mathcal{N}=2$ open string theory [35]. Since this theory [33] generalizes the Wess-Zumino-Witten-type model [36] for SDYM theory and thus describes only self-dual gauge fields (having helicity $+1$), it is not Lorentz invariant. Its maximally supersymmetric extension $\mathcal{N}=4$ super SDYM theory, however, does admit a Lorentz-invariant formulation [21, 37]. This theory features pairs of fields of opposite helicity. In [38], it was proposed to lift the corresponding Lagrangian to SFT and in [15], the twistor description of this model was given as a specialization of Witten's supertwistor SFT when one allows the string to vibrate only in part of the supertwistor space (not in $\mathbb{C}P^1 \hookrightarrow \mathcal{P}^{3|4}$). The form of the matrix model action given by (3.34) is identical to an action of this cubic string field theory for open $\mathcal{N}=2$ strings [15]. Let us comment on that point in more detail.

First of all, recall the definition of cubic open string field theory [39]. Take a $\mathbb{Z}$-graded algebra $\mathfrak{A}$ with an associative product $\star$ and a derivative Q with $Q^2=0$ and $|Q\mathcal{A}| = |\mathcal{A}|+1$ for any $\mathcal{A} \in \mathfrak{A}$. Assume furthermore a map $\int : \mathfrak{A} \to \mathbb{C}$ which gives non-vanishing results only for elements of grading 3 and respects the grading, i.e. $\int \mathcal{A} \star \mathcal{B} = (-1)^{|\mathcal{A}||\mathcal{B}|} \int \mathcal{B} \star \mathcal{A}$. The (formal) action of cubic string field theory is then

$$S = \frac{1}{2} \int \left(\mathcal{A} \star Q\mathcal{A} + \frac{2}{3} \mathcal{A} \star \mathcal{A} \star \mathcal{A} \right) \ . \tag{3.35}$$

The action is invariant under the gauge transformations $\delta\mathcal{A} = Q\varphi - \varphi \star \mathcal{A} + \mathcal{A} \star \varphi$. One can easily extend this action to allow for Chan-Paton factors by replacing $\mathfrak{A}$ with $\mathfrak{A} \otimes \mathrm{gl}(n, \mathbb{C})$ and $\int$ with $\int \otimes \mathrm{tr}$.

The physical interpretation of the above construction is the following: $\mathcal{A}$ is a "string field" encoding all possible excitations of an open string. The operator $\star$ glues the halves of two open strings to form a third one, and the operator $\int$ folds an open string and glues its two halves together [39].

To qualify as a string field, $\mathcal{A}$ is a functional of the embedding map Φ from the string parameter space to the string target space. For the case at hand, we take

$$\Phi : [0, \pi] \times G \ \to \ \mathcal{P}^{3|4}_{\varepsilon} \ , \tag{3.36}$$

where $\sigma \in [0, \pi]$ parameterizes the open string and $G \ni v$ provides the appropriate set of Graßmann variables on the worldsheet. Expanding $\Phi(\sigma, v) = \phi(\sigma) + v\psi(\sigma)$, this map embeds the $\mathcal{N}=2$ spinning string into supertwistor space. Next, we recollect $\phi = (z^\alpha = x^{\alpha\dot\alpha}\lambda_{\dot\alpha}, \eta_i = \eta_i^{\dot\alpha}\lambda_{\dot\alpha}, \lambda, \bar\lambda)$ and allow the string to vibrate only in the z^α-directions but keep the G-even zero modes of $(\eta_i, \lambda, \bar\lambda)$, so that the string field depends on $\{z^\alpha(\sigma), \eta_i, \lambda, \bar\lambda; \psi^{\alpha\ddot\alpha}(\sigma)\}$

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only [15]. Note that with ψ and η, we have two types of fermionic fields present, since we are implicitly working in the doubly supersymmetric description of superstrings [40], which we will briefly discuss in section 4.2. Therefore, the two fermionic fields are linked via a superembedding condition. We employ a suitable BRST operator $Q = \bar{D} + \bar{\partial}$, where $\bar{D} = \psi^{\alpha\bar{1}}\lambda^{\dot{\alpha}}\partial_\sigma x_{\alpha\dot{\alpha}} \in \mathcal{O}(1)$ and $\bar{\partial} \in \mathcal{O}(0)$ are type $(0,1)$ vector fields on the fibres and the base of $\mathcal{P}^{3|4}_\varepsilon$, respectively, and split the string field accordingly, $\mathcal{A} = \mathcal{A}_{\bar{D}} + \mathcal{A}_{\bar{\partial}}$. With a holomorphic integration measure on $\mathcal{P}^{3|4}_\varepsilon$, the Chern-Simons action (2.38) projects to [15]

$$S \;=\; \int_{\mathcal{Y}_\varepsilon} \omega \wedge \langle\, \mathrm{tr}\,(\mathcal{A}_{\bar{D}} \star \bar{\partial}\mathcal{A}_{\bar{D}} + 2\mathcal{A}_{\bar{D}} \star \bar{D}\mathcal{A}_{\bar{\partial}} + 2\mathcal{A}_{\bar{\partial}} \star \mathcal{A}_{\bar{D}} \star \mathcal{A}_{\bar{D}}\rangle\ . \tag{3.37}$$

Note that the string fields $\mathcal{A}_{\bar{D}}$ and $\mathcal{A}_{\bar{\partial}}$ are fermionic, i.e. they behave in the action as if they were forms multiplied with the wedge product. Furthermore, the above-mentioned $\mathbb{Z}$-grading of all the ingredients of this action has to be adjusted appropriately. Giving an expansion in η_i for these string fields similar to the one in (2.43a) and (2.43b), one recovers the super string field theory proposed by Berkovits and Siegel [38]. Its zero modes describe self-dual $\mathcal{N} = 4$ SDYM theory.

By identifying $Q+\mathcal{A}_Q$ with $\hat{\mathcal{X}}$, $\bar{\partial}$ with $\bar{\partial}_{\bar{\lambda}}$ and $\mathcal{A}_{\bar{\partial}}$ with $\hat{\mathcal{A}}^{0,1}_\Sigma$ and adjusting the $\mathbb{Z}_2$-grading of the fields, one obtains the action[10] (3.34) from (3.37). Therefore, we can e.g. translate solution generating techniques which are at hand for our matrix model immediately to the string field theory (3.37).

4. Identification with D-brane configurations

This section is devoted to presenting an interpretation of our matrix models in terms of D-brane configurations in superstring theory. After briefly reviewing some aspects of ordinary D-branes and super D-branes, we will relate the matrix models arising from hCS theory to topological D-branes. It follows the interpretation of the corresponding matrix model arising from SDYM theory as D-branes within $\mathcal{N} = 2$ string theory. Then we will switch to ten-dimensional $\mathcal{N} = 1$ superstrings and find a connection of our matrix models with a supersymmetric version of the ADHM construction. By adding a term to the matrix model action (3.13), we can even map all ingredients of the D-brane interpretation of the ADHM prescription from the moduli space $\mathbb{R}^{4|8}$ to the twistor space $\mathcal{P}^{3|4}_\varepsilon$. Eventually, we comment on the matrix models obtained from noncommutativity.

Note that we use different conventions in $\mathcal{N} = 1$ and $\mathcal{N} = 2$ critical superstring theories: The worldvolume of a Dp-brane is meant to have dimension $(1, p)$ and (a, b) with $a + b = p$, respectively.

4.1 Review of ordinary D-branes within D-branes

In type IIB superstring theory, the Ramond-Ramond sector contains i-form fields $C_{(i)}$ for $i = 0, 2, 4, 6, 8, 10$, which couple naturally to D-branes of spatial dimension $i - 1$. Recall that there are two different points of view for these D-branes. First, one can understand a Dp-brane as a p-dimensional hyperplane on which open strings end. Second, a Dp-brane is a soliton of type IIB supergravity in ten dimensions.

[10]Note that $\bar{\partial}Q + Q\bar{\partial} = 0$.

A stack of n Dp-branes naturally comes with a rank n vector bundle E over their $p+1$-dimensional worldvolume together with a connection one-form A. This field arises from the Chan-Paton factors attached to the ends of an open string. The equations determining the dynamics of A at low energies in a flat background are just the $\mathcal{N}=1$ super Yang-Mills equations with gauge group $\mathrm{U}(n)$, dimensionally reduced from ten to $p+1$ dimensions. The emerging Higgs fields determine the movement of the Dp-brane in its normal directions. On Kähler manifolds, the BPS sector is given by (a supersymmetric extension of) the Hermitean Yang-Mills equations[11]

$$F^{0,2} \;=\; 0 \;=\; F^{2,0} \quad \text{and} \quad k^{d-1}\wedge F \;=\; \gamma k^d \;, \tag{4.1}$$

which are also reduced appropriately from ten to $p+1$ dimensions, see e.g. [41]. Here, k is the Kähler form of the target space and γ is the slope of E, i.e. a constant enconding information about the first Chern class of the vector bundle E. These equations imply the (dimensionally reduced, supersymmetric) Yang-Mills equations.

If we just consider the topological subsector of the theory, the dynamics of the connection one-form A is described by an appropriate dimensional reduction of the holomorphic Chern-Simons equations [42 − 44], which are given by

$$F^{0,2} \;=\; 0 \;=\; F^{2,0} \;. \tag{4.2}$$

Thus, the dynamics of topological D-branes differs from the one of their BPS-cousins only by the second equation in (4.1), which is a stability condition on the vector bundle E.

A bound state of a stack of Dp-branes with a D(p-4)-brane can be described in two different ways. On the one hand, we can look at this state from the perspective of the higher-dimensional Dp-brane. Here, we find that the D(p-4) brane is described by a gauge field strength F on the bundle E over the worldvolume of the Dp-brane with a nontrivial second Chern character $ch_2(E)$. The instanton number (the number of D(p-4) branes) is given by the corresponding second Chern class. In particular, the BPS bound state of a stack of D3-branes with a D(-1)-brane is given by a self-dual field strength $F=*F$ on E with $-\frac{1}{8\pi^2}\int F\wedge F=1$. On the other hand, one can adopt the point of view of the D(p-4)-brane inside the Dp-brane and consider the dimensional reduction of the $\mathcal{N}=1$ super Yang-Mills equations from ten dimensions to the worldvolume of the D(p-4)-branes. To complete the picture, one has to add strings with one end on the Dp-brane and the other one on the D(p-4)-branes. Furthermore, one has to take into account that the presence of the Dp-brane will halfen the number of supersymmetries once more, usually to a chiral subsector. In the case of the above example of D3- and D(-1)-branes, this will give rise to the ADHM equations discussed later. The situation for bound states of Dp- and D(p-2)-branes can be discussed analogously, and a bound state of stacks of D3- and D1-branes will – from the perspective of the D1-branes – yield the Nahm equations.

[11] or "generalized Hitchin equations"

4.2 Super D-branes

As both the target space $\mathcal{P}_\varepsilon^{3|4}$ of the topological B-model and the corresponding moduli space $\mathbb{R}^{4|8}$ are supermanifolds, we are naturally led to consider D-branes which have also fermionic worldvolume directions.

Recall that there are three approaches of embedding worldvolumes into target spaces when Graßmann directions are involved. First, one has the Ramond-Neveu-Schwarz (RNS) formulation [45, 46], which maps a super worldvolume to a bosonic target space. This approach only works for a spinning particle and a spinning string; no spinning branes have been constructed so far. However, this formulation allows for a covariant quantization. Second, there is the Green-Schwarz (GS) formulation [47], in which a bosonic worldvolume is mapped to a target space which is a supermanifold. In this approach, the well-known κ-symmetry appears as a local worldvolume fermionic symmetry. Third, there is the doubly-supersymmetric formulation (see [40] and references therein), which unifies in some sense both the RNS and GS approaches. In this formulation, an additional superembedding condition is imposed, which reduces the worldvolume supersymmetry to the κ-symmetry of the GS approach.

In the following, we will always work implicitly with the doubly supersymmetric approach.

4.3 Topological D-branes and the matrix models

The interpretation of the matrix model (3.13) is now rather straightforward. For gauge group $\mathrm{GL}(n,\mathbb{C})$, it describes a stack of n almost space-filling D(1|4)-branes, whose fermionic dimensions only extend into the holomorphic directions of the target space $\mathcal{P}_\varepsilon^{3|4}$. These D-branes furthermore wrap a $\mathbb{C}P_x^{1|4}\hookrightarrow\mathcal{P}_\varepsilon^{3|4}$.

We can use the expansion $\mathcal{X}_\alpha = \mathcal{X}_\alpha^0 + \mathcal{X}_\alpha^i\eta_i + \mathcal{X}_\alpha^{ij}\eta_i\eta_j + \dots$ on any patch of $\mathbb{C}P^{1|4}$ to examine the equations of motion (2.41a) more closely:

$$\begin{aligned}
[\mathcal{X}_1^0,\mathcal{X}_2^0] &= 0\,,\\
[\mathcal{X}_1^i,\mathcal{X}_2^0]+[\mathcal{X}_1^0,\mathcal{X}_2^i] &= 0\,,\\
\{\mathcal{X}_1^i,\mathcal{X}_2^j\}-\{\mathcal{X}_1^j,\mathcal{X}_2^i\}+[\mathcal{X}_1^{ij},\mathcal{X}_2^0]+[\mathcal{X}_1^0,\mathcal{X}_2^{ij}] &= 0\,,\\
&\dots
\end{aligned}\tag{4.3}$$

Clearly, the bodies $\mathcal{X}_\alpha^0$ of the Higgs fields can be diagonalized simultaneously, and the diagonal entries describe the position of the D(1|4)-brane in the normal directions of the ambient space $\mathcal{P}_\varepsilon^{3|4}$. In the fermionic directions, this commutation condition is relaxed and thus, the D-branes can be smeared out in these directions even in the classical case.

4.4 Interpretation within $\mathcal{N}=2$ string theory

The critical $\mathcal{N}=2$ string has a four-dimensional target space and its open string effective field theory is self-dual Yang-Mills theory (or its noncommutative deformation [48] in the presence of a B-field). It has been argued [21] that, after extending the $\mathcal{N}=2$ string effective action in a natural way to recover Lorentz invariance, the effective field theory becomes the full $\mathcal{N} = 4$ supersymmetrically extended SDYM theory, and we will adopt this point of view in the following.

Considering D-branes in this string theory is not as natural as in ten-dimensional superstring theories since the NS sector is connected to the R sector via the $\mathcal{N} = 2$ spectral flow, and it is therefore sufficient to consider the purely NS part of the $\mathcal{N} = 2$ string. Nevertheless, one can confine the end-points of the open strings in this theory to certain subspaces and impose Dirichlet boundary conditions to obtain objects which we will call D-branes in $\mathcal{N} = 2$ string theory. Altough the meaning of these objects has not yet been completely established, there seem to be a number of safe statements we can recollect. First of all, the effective field theory of these D-branes is four-dimensional (supersymmetric) SDYM theory reduced to the appropriate worldvolume [49, 50]. The four-dimensional SDYM equations are nothing but the Hermitean Yang-Mills equations:

$$F^{2,0} \;=\; F^{0,2} \;=\; 0 \quad \text{and} \quad k \wedge F \;=\; 0\ , \tag{4.4}$$

where k is again the Kähler form of the background. The Higgs fields arising in the reduction process describe again fluctuations of the D-branes in their normal directions.

As is familiar from the topological models yielding hCS theory, we can introduce A- and B-type boundary conditions for the D-branes in $\mathcal{N} = 2$ critical string theory. For the target space $\mathbb{R}^{2,2}$, the A-type boundary conditions are compatible with D-branes of worldvolume dimension (0,0), (0,2), (2,0) and (2,2) only [51, 50].

Thus, we find a first interpretation of our matrix model (3.1) in terms of a stack of n D0- or D(0|8)-branes in $\mathcal{N} = 2$ string theory, and the topological D(1|4)-brane is the equivalent configuration in B-type topological string theory.

As usual, turning on a B-field background will give rise to noncommutative deformations of the ambient space, and therefore the matrix model (3.34) describes a stack of n D4-branes in $\mathcal{N} = 2$ string theory within such a background.

The moduli superspaces $\mathbb{R}^{4|8}_\theta$ and $\mathbb{R}^{0|8}$ for both the noncommutative and the ordinary matrix model can therefore be seen as *chiral* D(4|8)- and D(0|8)-branes, respectively, with $\mathcal{N} = 4$ self-dual Yang-Mills theory as the appropriate (chiral) low energy effective field theory.

4.5 ADHM equations and D-branes

The ADHM algorithm [52] for constructing instanton solutions has found a nice interpretation in the context of string theory [53]; see also [54] and [55] for a helpful review. We will follow the discussion of the latter reference and start from a configuration of k D5-branes bound to a stack of n D9-branes, which – upon dimensional reduction – will eventually yield a configuration of k D(-1)-branes inside a stack of n D3-branes.

From the perspective of the D5-branes, the $\mathcal{N} = 2$ supersymmetry of type IIB superstring theory is broken down to $\mathcal{N} = (1,1)$ on the six-dimensional worldvolume of the D5-branes, which are BPS. The fields in the ten-dimensional Yang-Mills multiplet are rearranged into an $\mathcal{N} = 2$ vector multiplet $(\phi_a, A_{\alpha\dot\alpha}, \chi^i_\alpha, \bar\mu^{\dot\alpha}_i)$, where the indices $i = 1, \dots, 4$,

$a = 1, \ldots 6$ and $\alpha, \dot{\alpha} = 1, 2$ label the representations of the Lorentz group $\mathrm{SO}(5,1) \sim \mathrm{SU}(4)$ and the R-symmetry group $\mathrm{SO}(4) \sim \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, respectively. Thus, ϕ and A denote bosons, while χ and $\bar{\mu}$ refer to fermionic fields. Note that the presence of the D9-branes will further break supersymmetry down to $\mathcal{N} = (0,1)$ and therefore the above multiplet splits into the vector multiplet $(\phi_a, \bar{\mu}_i^{\dot{\alpha}})$ and the hypermultiplet $(A_{\alpha\dot{\alpha}}, \chi_\alpha^i)$. In the following, we will discuss the field theory on the D5-branes in the language of $\mathcal{N} = (0,1)$ supersymmetry.

Let us now consider the vacuum moduli space of this theory which is called the Higgs branch. This is the sector of the theory, where the D-field, i.e. the auxiliary field for the $\mathcal{N} = (0,1)$ vector multiplet, vanishes[12]. From the Yang-Mills part describing the vector multiplet, we have the contribution $4\pi^2\alpha'^2 \int \mathrm{d}^6x \, \mathrm{tr}_k \frac{1}{2} D_{\mu\nu}^2$, where we also introduce the notation $D_{\mu\nu} = \mathrm{tr}_2(\vec{\sigma}\bar{\sigma}_{\mu\nu}) \cdot \vec{D}$. The hypermultiplet leads to an additional contribution of $\int \mathrm{d}^6x \, \mathrm{tr}_k \mathrm{i}\vec{D} \cdot \vec{\sigma}^{\dot{\alpha}}{}_{\dot{\beta}} \bar{A}^{\alpha\dot{\beta}} A_{\alpha\dot{\alpha}}$. Note that we use a bar instead of the dagger to simplify notation. However, this bar must not be confused with complex conjugation.

It remains to include the contributions from open strings having one end on a D5-brane and the other one on a D9-brane. These additional degrees of freedom form two hypermultiplets under $\mathcal{N} = (0,1)$ supersymmetry, which sit in the bifundamental representation of $\mathrm{U}(k) \times \mathrm{U}(n)$ and its conjugate. We denote them by $(w_{\dot{\alpha}}, \psi^i)$ and $(\bar{w}^{\dot{\alpha}}, \bar{\psi}^i)$, where $w_{\dot{\alpha}}$ and $\bar{w}^{\dot{\alpha}}$ and ψ^i and $\bar{\psi}^i$ denote four complex scalars and eight Weyl spinors, respectively. The contribution to the D-terms is similar to the hypermultiplet considered above: $\int \mathrm{d}^6x \, \mathrm{tr}_k \mathrm{i}\vec{D} \cdot \vec{\sigma}^{\dot{\alpha}}{}_{\dot{\beta}} \bar{w}^{\dot{\beta}} w_{\dot{\alpha}}$.

Collecting all the contributions of the D-field to the action and varying them yields the equations of motion

$$\alpha'^2 \vec{D} \;=\; \frac{\mathrm{i}}{16\pi^2} \vec{\sigma}^{\dot{\alpha}}{}_{\dot{\beta}} (\bar{w}^{\dot{\beta}} w_{\dot{\alpha}} + \bar{A}^{\alpha\dot{\beta}} A_{\alpha\dot{\alpha}}) \; . \tag{4.5}$$

After performing the dimensional reduction of the D5-brane to a D(-1)-brane, the condition that $\vec{D}$ vanishes is equivalent to the ADHM constraints.

Spelling out all possible indices on our fields, we have $A_{\alpha\dot{\alpha}pq}$ and $w_{up\dot{\alpha}}$, where $p, q = 1, \ldots, k$ denote indices of the representation $\mathbf{k}$ of the gauge group $\mathrm{U}(k)$ while $u = 1, \ldots, n$ belongs to the $\mathbf{n}$ of $\mathrm{U}(n)$. Let us introduce the new combinations of indices $r = u + p \otimes \alpha = 1, \ldots, n + 2k$ together with the matrices

$$(a_{rq\dot{\alpha}}) \;=\; \begin{pmatrix} w_{uq\dot{\alpha}} \\ A_{\alpha\dot{\alpha}pq} \end{pmatrix} \; , \quad (\bar{a}_q^{\dot{\alpha}r}) \;=\; \begin{pmatrix} \bar{w}_{qu}^{\dot{\alpha}} & A_{pq}^{\alpha\dot{\alpha}} \end{pmatrix} \quad \text{and} \quad (b_{rq}^{\beta}) \;=\; \begin{pmatrix} 0 \\ \delta_\alpha{}^\beta \delta_{pq} \end{pmatrix} \; , \tag{4.6}$$

which are of dimension $(n+2k) \times 2k$, $2k \times (n+2k)$ and $(n+2k) \times 2k$, respectively. Now we are ready to define a $(n+2k) \times 2k$ dimensional matrix, the zero-dimensional Dirac operator of the ADHM construction, which reads

$$\Delta_{rp\dot{\alpha}}(x) \;=\; a_{rp\dot{\alpha}} + b_{rp}^{\alpha} x_{\alpha\dot{\alpha}} \; , \tag{4.7}$$

and we put $\bar{\Delta}_p^{\dot{\alpha}r} := (\Delta_{rp\dot{\alpha}})^*$. Written in the new components (4.6), the ADHM constraints amounting to the D-flatness condition read $\vec{\sigma}^{\dot{\alpha}}{}_{\dot{\beta}} (\bar{a}^{\dot{\beta}} a_{\dot{\alpha}}) = 0$, or, more explicitly,

$$\bar{a}_{\dot{\alpha}} a_{\dot{\beta}} + \bar{a}_{\dot{\beta}} a_{\dot{\alpha}} \;=\; 0 \; , \tag{4.8}$$

[12] This is often referred to as the D-flatness condition.

where we defined as usual $\bar{a}_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar{a}^{\dot\beta}$. All further conditions, which are sometimes also summarized under ADHM constraints, are automatically satisfied due to our choice of b^{α}_{rp} and the reality properties of our fields.

The kernel of the zero-dimensional Dirac operator is generally of dimension n, as this is the difference between its numbers of rows and columns. It is spanned by vectors which can be arranged to a complex matrix U_{ru} satisfying

$$\bar{\Delta}^{\dot\alpha r}_{p} U_{ru} \;=\; 0 \; . \tag{4.9}$$

Upon demanding that the frame U_{ru} is orthonormal, i.e. that $\bar{U}^{r}_{u} U_{rv} = \delta_{uv}$, we can construct a self-dual $\mathrm{SU}(n)$-instanton configuration from

$$(\mathscr{A}_{\alpha\dot\alpha})_{uv} \;=\; \bar{U}^{r}_{u}\partial_{\alpha\dot\alpha} U_{rv} \; . \tag{4.10}$$

Usually, one furthermore introduces the auxiliary matrix f via

$$f \;=\; 2(\bar{w}^{\dot\alpha} w_{\dot\alpha} + (A_{\alpha\dot\alpha} + x_{\alpha\dot\alpha} \otimes \mathbb{1}_k)^2)^{-1} \; , \tag{4.11}$$

which fits in the factorization condition $\bar{\Delta}^{\dot\alpha r}_{p}\Delta_{rq\dot\beta} = \delta^{\dot\alpha}_{\dot\beta}(f^{-1})_{pq}$. Note that the latter condition is again equivalent to the ADHM constraints (4.8) arising from (4.5). The matrix f allows for an easy computation of the field strength

$$\mathscr{F}_{\mu\nu} \;=\; 4\bar{U} b \sigma_{\mu\nu} f \bar{b} U \tag{4.12}$$

and the instanton number

$$k \;=\; -\frac{1}{16\pi^2}\int \mathrm{d}^4x \,\mathrm{tr}_{\,n} \mathscr{F}^2_{\mu\nu} \;=\; \frac{1}{16\pi^2}\int \mathrm{d}^4x \Box^2 \,\mathrm{tr}_{\,k} \log f \; . \tag{4.13}$$

The self-duality of $\mathscr{F}_{\mu\nu}$ in (4.12) is evident from the self-duality property of $\sigma_{\mu\nu}$.

4.6 Super ADHM construction and super D-branes

Recall that there is a formulation of the ordinary super Yang-Mills equations and their self-dual truncations in terms of superfields, which we already used e.g. in (2.47). In the superformulation, the field content and the equations of motion take the same shape as in the ordinary formulation, but with all the fields being superfields. Moreover, one can find an Euler operator, which easily shows the equivalence of the superfield equations with the ordinary field equations, see e.g. [56, 20].

For the super ADHM construction, let us consider k D(5|8)-branes inside n D(9|8)-branes. To describe this scenario, it is only natural to extend the fields arising from the strings in this configuration to superfields on $\mathbb{C}^{10|8}$ and the appropriate subspaces. In particular, we extend the fields $w_{\dot\alpha}$ and $A_{\alpha\dot\alpha}$ entering into the bosonic D-flatness condition to superfields living on $\mathbb{C}^{6|8}$. However, since supersymmetry is broken down to four copies of $\mathcal{N} = 1$ due to the presence of the two stacks of D-branes, these superfields can only be linear in the Graßmann variables. From the discussion in [56], we can then state what the superfield expansion should look like:

$$w_{\dot\alpha} \;=\; \overset{\circ}{w}_{\dot\alpha} + \psi^i \eta_{i\dot\alpha} \quad \text{and} \quad A_{\alpha\dot\alpha} \;=\; \overset{\circ}{A}_{\alpha\dot\alpha} + \chi^i_{\alpha}\eta_{i\dot\alpha} \; . \tag{4.14}$$

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After following the above discussion, we arrive at the D-flatness condition

$$\alpha'^2 \vec{D} \;=\; \frac{\mathrm{i}}{16\pi^2}\vec{\sigma}^{\dot\alpha}{}_{\dot\beta}(\bar{w}^{\dot\beta}w_{\dot\alpha} + \bar{A}^{\alpha\dot\beta}A_{\alpha\dot\alpha}) \;=\; 0\;, \tag{4.15}$$

where now all fields are true superfields. After performing the dimensional reduction of the D(9|8)-D(5|8)-brane configuration to one containing D(3|8)- and D(-1|8)-branes and arranging the resulting field content according to (4.6), we can construct the zero-dimensional super Dirac operator[13]

$$\Delta_{ri\dot\alpha} \;=\; a_{ri\dot\alpha} + b^{\alpha}_{ri}x^R_{\alpha\dot\alpha} \;=\; \mathring{a}_{ri\dot\alpha} + b^{\alpha}_{ri}x^R_{\alpha\dot\alpha} + c^j_{ri}\eta_{j\dot\alpha}\;, \tag{4.16}$$

where $(x^{\alpha\dot\alpha}_R, \eta^{\dot\alpha}_i)$ are coordinates on the (anti-)chiral superspace $\mathbb{C}^{4|8}$. That is, from the point of view of the full superspace $\mathbb{C}^{4|16}$ with coordinates $(x^{\alpha\dot\alpha}, \theta^{i\alpha}, \eta^{\dot\alpha}_i)$, we have $x^{\alpha\dot\alpha}_R = x^{\alpha\dot\alpha} + \theta^{i\alpha}\eta^{\dot\alpha}_i$. The super ADHM constraints (4.15) were discussed in [57] for the first time; see also [58] for a related recent discussion.

In components, these super constraints (4.8) read

$$\bar{\mathring{a}}_{\dot\alpha}\mathring{a}_{\dot\beta} + \bar{\mathring{a}}_{\dot\beta}\mathring{a}_{\dot\alpha} \;=\; 0\;,\quad \bar{\mathring{a}}_{\dot\alpha}c_i - \bar{c}_i\mathring{a}_{\dot\alpha} \;=\; 0\;,\quad \bar{c}_i c_j - \bar{c}_j c_i \;=\; 0\;. \tag{4.17}$$

The additional sign in the equations involving c_i arises from ordering and extracting the Graßmann variables $\eta^{\dot\alpha}_i$ as well as the definition $\overline{c_i\eta^{\dot\alpha}_i} = \eta^{\dot\alpha}_i\bar{c}_i = -\bar{c}_i\eta^{\dot\alpha}_i$.

As proven in [57], this super ADHM construction gives rise to solutions to the $\mathcal{N}=4$ self-dual Yang-Mills equations in the form of the super gauge potentials

$$\mathscr{A}_{\alpha\dot\alpha} \;=\; \bar{U}\partial_{\alpha\dot\alpha}U \quad\text{and}\quad \mathscr{A}^i_{\dot\alpha} \;=\; \bar{U}D^i_{\dot\alpha}U\;, \tag{4.18}$$

where U and $\bar{U}$ are again zero modes of $\bar\Delta$ and Δ, normalized according to $\bar{U}U = \mathbb{1}$. That is, the super gauge potentials in (4.18) satisfy the self-duality equations (2.47).

The fact that solutions to the $\mathcal{N}=4$ SDYM equations in general do not satisfy the $\mathcal{N}=4$ SYM equations does not spoil our interpretation of such solutions as D(-1|8)-branes, since in our picture, $\mathcal{N}=4$ supersymmetry is broken down to four copies of $\mathcal{N}=1$ supersymmetry. Note furthermore that $\mathcal{N}=4$ SYM theory and $\mathcal{N}=4$ SDYM theory can be seen as different weak coupling limits of *one* underlying field theory [1].

4.7 The SDYM matrix model and the super ADHM construction

While a solution to the $\mathcal{N}=4$ SDYM equations with gauge group U(n) and second Chern number $c_2 = k$ describes a bound state of k D(-1|8)-branes with n D(3|8)-branes at low energies, the SDYM matrix model obtained by a dimensional reduction of this situation describes a bound state between $k+n$ D(-1|8)-branes. This implies that there is only one type of strings, i.e. those having both ends on the D(-1|8)-branes. In the ADHM construction, one can simply account for this fact by eliminating the bifundamental fields, i.e. by putting $w_{\dot\alpha}$ and ψ^i to zero.

[13] One should stress, that an extension of the Dirac operator to higher orders in the Graßmann variables is inconsistent with the ADHM construction, as is easily seen from its original motivation via monads. The same is suggested from the supersymmetries present in our D-brane configuration.

Hence, the remaining ADHM constraints read

$$\vec{\sigma}^{\dot{\alpha}}{}_{\dot{\beta}}(\bar{A}^{\alpha\dot{\beta}}A_{\alpha\dot{\alpha}}) \; = \; 0 \; , \tag{4.19}$$

and one can use the reality conditions to show that these equations are equivalent to

$$\varepsilon^{\alpha\beta}[A_{\alpha\dot{\alpha}}, A_{\beta\dot{\beta}}] \; = \; 0 \; . \tag{4.20}$$

The expansion (4.14) yields

$$[A_{\alpha\dot{\alpha}}, \chi^{\alpha i}] \; = \; 0 \; . \tag{4.21}$$

Thus, we recover the matrix SDYM equations (3.2) with fields of higher R charges put to zero. This is expected since fields with more than one R-symmetry index appear beyond linear order in the Graßmann fields, and their presence would spoil the ADHM construction.

4.8 Extension of the matrix model

It is now conceivable that the D3-D(-1)-brane[14] system explaining the ADHM construction can be carried over to the supertwistor space $\mathcal{P}^{3|4}_\varepsilon$. To this end, we take a D1-D5-brane system and analyze it, either via open D5-D5 strings with excitations corresponding in the holomorphic Chern-Simons theory to gauge configurations with non-trivial second Chern character, or else by looking at the D1-D1 and the D1-D5 strings. The latter point of view gives rise to a holomorphic Chern-Simons analogue of the ADHM configuration, as we will show in the following.

The action for the D1-D1 strings is evidently our hCS matrix model (3.13). To incorporate the D1-D5 strings, we can use an action proposed by Witten in [1][15]

$$\int_{\mathcal{Y}_\varepsilon} \omega \wedge \mathrm{tr}\,(\beta\bar{\partial}\alpha + \beta\mathcal{A}^{0,1}_\Sigma\alpha) \; , \tag{4.22}$$

where the fields α and β take values in the line bundles $\mathcal{O}(1)$ and transform in the fundamental and antifundamental representation of the gauge group $\mathrm{GL}(n,\mathbb{C})$, respectively.

The equations of motion of the total matrix model, whose action is the sum of (3.13) and (4.22), are then modified to

$$\begin{aligned} \bar{\partial}\mathcal{X}_\alpha + [\mathcal{A}^{0,1}_\Sigma, \mathcal{X}_\alpha] \; &= \; 0 \; , \\ [\mathcal{X}_1, \mathcal{X}_2] + \alpha\beta \; &= \; 0 \; , \\ \bar{\partial}\alpha + \mathcal{A}^{0,1}_\Sigma\alpha \; = \; 0 \quad \text{and} \quad \bar{\partial}\beta + \beta\mathcal{A}^{0,1}_\Sigma \; &= \; 0 \; . \end{aligned} \tag{4.23}$$

Similarly to the Higgs fields $\mathcal{X}_\alpha$ and the gauge potential $\mathcal{A}^{0,1}_\Sigma$, we can give a general field expansion for β and $\alpha = \bar{\beta}$:

$$\begin{aligned} \beta_+ \; &= \lambda^{\dot{\alpha}}_+ w_{\dot{\alpha}} + \psi^i\eta^+_i + \gamma_+\frac{1}{2!}\eta^+_i\eta^+_j\hat{\lambda}^{\dot{\alpha}}_+\rho^{ij}_{\dot{\alpha}} + \gamma^2_+\frac{1}{3!}\eta^+_i\eta^+_j\eta^+_k\hat{\lambda}^{\dot{\alpha}}_+\hat{\lambda}^{\dot{\beta}}_+\sigma^{ijk}_{\dot{\alpha}\dot{\beta}} \\ &\quad + \gamma^3_+\frac{1}{4!}\eta^+_i\eta^+_j\eta^+_k\eta^+_l\hat{\lambda}^{\dot{\alpha}}_+\hat{\lambda}^{\dot{\beta}}_+\hat{\lambda}^{\dot{\gamma}}_+\tau^{ijkl}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} \; , \\ \alpha_+ \; &= \lambda^{\dot{\alpha}}_+\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{w}^{\dot{\beta}}_+ + \bar{\psi}^i\eta^+_i + \dots \; . \end{aligned} \tag{4.24}$$

[14]For simplicity, let us suppress the fermionic dimensions of the D-branes in the following.

[15]In fact, he uses this action to complement the hCS theory in such a way that it gives rise to full Yang-Mills theory on the moduli space. For this, he changes the parity of the fields α and β to be fermionic.

Applying the equations of motion, one learns that the fields beyond linear order in the Graßmann variables are composite fields:

$$\rho^{ij}_{\dot\alpha} \;=\; w_{\dot\alpha}\phi^{ij}\ ,\quad \sigma^{ijk}_{\dot\alpha\dot\beta} \;=\; \frac{1}{2}w_{(\dot\alpha}\tilde\chi^{ijk}_{\dot\beta)} \quad\text{and}\quad \tau^{ijkl}_{\dot\alpha\dot\beta\dot\gamma} \;=\; \frac{1}{3}w_{(\dot\alpha}G^{ijkl}_{\dot\beta\dot\gamma)}\ . \tag{4.25}$$

We intentionally denoted the zeroth order components of α_+ and β_+ by $\lambda^+_{\dot\alpha}\bar w^{\dot\alpha}$ and $\lambda^{\dot\alpha}_+ w_{\dot\alpha}$, respectively, since the expansion (4.24) together with the field equations (4.23) are indeed the (super) ADHM equations

$$\vec\sigma^{\dot\alpha}{}_{\dot\beta}(\bar w^{\dot\beta}w_{\dot\alpha} + A^{\alpha\dot\alpha}A_{\alpha\dot\beta}) \;=\; 0\ , \tag{4.26}$$

which are equivalent to the condition that $\bar\Delta\Delta = \mathbb{1}_2\otimes f^{-1}$. Again, the components beyond linear order in the Graßmann fields have been put to zero in the Higgs fields $\mathcal{X}_\alpha$ and the gauge potential $\mathcal{A}^{0,1}_\Sigma$ (which automatically does the same for the fields α and β).

This procedure seems at first slightly ad-hoc, but again it becomes quite natural, when recalling that for the ADHM D-brane configuration, supersymmetry is broken from $\mathcal{N}=4$ to four times $\mathcal{N}=1$. Furthermore, the fields which are put to zero give rise to the potential terms in the action, and thus, we can regard putting these fields to zero as an additional "D-flatness condition" arising on the topological string side.

With this additional constraint, our matrix model (3.13) together with the extension (4.22) is equivalent to the ADHM equations. Therefore, it is dual to holomorphic Chern-Simons theory on the full supertwistor space $\mathcal{P}^{3|4}_\varepsilon$, in the same sense in which the ADHM construction is dual to SDYM theory.

Summarizing, the D3-D(-1)-brane system can be mapped via an extended Penrose-Ward-transform to a D5-D1-brane system in topological string theory. The arising super SDYM theory on the D3-brane corresponds to hCS theory on the D5-brane, while the matrix model describing the effective action on the D(-1)-brane corresponds to our hCS matrix model on a topological D1-brane. The additional D3-D(-1) strings completing the picture from the perspective of the D(-1)-brane can be directly translated into additional D5-D1 strings on the topological side. The ADHM equations can furthermore be obtained from an extension of the hCS matrix model on the topological D1-brane with a restriction on the field content.

4.9 D-branes in a nontrivial B-field background

Except for the remarks on the $\mathcal{N}=2$ string, we have not yet discussed the matrix model which we obtained from deforming the moduli space $\mathbb{R}^{4|8}$ to a noncommutative spacetime.

In general, noncommutativity is interpreted as the presence of a Kalb-Ramond B-field background in string theory. Thus, solutions to the noncommutative SDYM theory (3.28) on $\mathbb{R}^{4|8}_\theta$ are D(-1|8)-branes bound to a stack of space-filling D(3|8)-branes in the presence of a B-field background. This distinguishes the commutative from the noncommutative matrix model: The noncommutative matrix model is now dual to the ADHM equations, instead of being embedded like the commutative one.

The matrix model on holomorphic Chern-Simons theory describes analogously a topological almost space-filling D(5|4)-brane in the background of a B-field. Note that a noncommutative deformation of the target space $\mathcal{P}^{3|4}_\varepsilon$ does not yield any inconsistencies in the context of the topological B-model. Such deformations have been studied e.g. in [59] and [41].

On the one hand, we found two pairs of matrix models, which are dual to each other (as the ADHM equations are dual to the SDYM equations). On the other hand, we expect both pairs to be directly equivalent to one another in a certain limit, in which the rank of the gauge group of the commutative matrix model tends to infinity. The implications of this observation might reveal some further interesting features.

5. Dimensional reductions related to the Nahm equations

After the discussion of the ADHM construction in the previous section, one is led to try to also translate the D-brane interpretation of the Nahm construction to some topological B-model on a Calabi-Yau supermanifold. This is in fact possible, but since the D-brane configuration is somewhat more involved, we will refrain from presenting details. In the subsequent discussion, we strongly rely on results from [11], where further details complementing our rather condensed presentation can be found. As in this reference, we will constrain our considerations to real structures yielding Euclidean signature, i.e. $\varepsilon = -1$.

5.1 The D-brane interpretation of the Nahm construction

Before presenting its super extension, let us briefly recollect the ordinary Nahm construction [60] starting from its D-brane interpretation [61] and [62]; see also [54]. For simplicity, we restrict ourselves to the case of SU(2)-monopoles, but a generalization of our discussion to gauge groups of higher rank is possible and rather straightforward.

We start in ten-dimensional type IIB superstring theory with a pair of D3-branes extended in the directions $1, 2, 3$ and located at $x^4 = \pm 1$, $x^M = 0$ for $M > 4$. Consider now a bound state of these D3-branes with k D1-branes extending along the x^4-axis and ending on the D3-branes. As in the case of the ADHM construction, we can look at this configuration from two different points of view.

From the perspective of the D3-branes, the effective field theory on their worldvolume is $\mathcal{N} = 4$ super Yang-Mills theory. The D1-branes bound to the D3-branes and ending on them impose a BPS condition, which amounts to the Bogomolny equations in three dimensions,

$$D_a \Phi \; = \; \frac{1}{2} \varepsilon_{abc} F_{bc} \; , \tag{5.1}$$

where $a, b, c = 1, 2, 3$. The ends of the D1-branes act as magnetic charges in the worldvolume of the D3-branes. They can therefore be understood as magnetic monopoles [63], whose field configuration (Φ, A_a) satisfies the Bogomolny equations. These monopoles are static solutions of the underlying Born-Infeld action.

From the perspective of the D1-branes, the effective field theory is first $\mathcal{N} = (8, 8)$ super Yang-Mills theory in two dimensions, but supersymmetry is broken by the presence of the two D3-branes to $\mathcal{N} = (4, 4)$. As before, one can write down the corresponding D-terms [54] and impose a D-flatness condition:

$$D = \frac{\partial X^a}{\partial x^4} + [A_4, X^a] - \frac{1}{2} \varepsilon_{abc} [X^b, X^c] + R \; = \; 0 \; , \tag{5.2}$$

where the X^a are the scalar fields corresponding to the directions in which the D3-branes extend. The R-term is proportional to $\delta(x^4 \pm 1)$ and allow for the D1-branes to end on the D3-branes. It is related to the so-called Nahm boundary conditions, which we do not discuss. The theory we thus found is simply self-dual Yang-Mills theory, reduced to one dimension. By imposing "temporal gauge" $A_4 = 0$, we arrive at the Nahm equations

$$\frac{\partial X^a}{\partial s} - \frac{1}{2}\varepsilon_{abc}[X^b, X^c] \;=\; 0 \quad \text{for} \quad -1 \;<\; s \;<\; 1 \;, \tag{5.3}$$

where we substituted $s = x^4$. From solutions to these (integrable) equations, we can construct the one-dimensional Dirac operator

$$\Delta^{\dot\alpha\dot\beta} \;=\; (\mathbb{1}_2)^{\dot\alpha\dot\beta} \otimes \frac{\partial}{\partial s} + \sigma_a^{(\dot\alpha\dot\beta)}(x^a - X^a) \;. \tag{5.4}$$

The equations (5.3) are, analogously to the ADHM equations, the condition for $\bar\Delta\Delta$ to commute with the Pauli matrices, or equivalently, to have an inverse f:

$$\bar\Delta\Delta \;=\; \mathbb{1}_2 \otimes f^{-1} \;. \tag{5.5}$$

The normalized zero modes U of the Dirac operator $\bar\Delta$ satisifying

$$\bar\Delta(s) U \;=\; 0 \;, \qquad \int_{-1}^{1} \mathrm{d}s \; \bar U(s) U(s) \;=\; \mathbb{1} \tag{5.6}$$

then give rise to solutions to the Bogomolny equations (5.1) via the definitions

$$\varPhi(x) \;=\; \int_{-1}^{1} \mathrm{d}s \; \bar U(s)\, s\, U(s) \quad \text{and} \quad \mathscr{A}_a(x) \;=\; \int_{-1}^{1} \mathrm{d}s \; \bar U(s) \frac{\partial}{\partial x^a} U(s) \;. \tag{5.7}$$

The verification of this statement is straightforward when using the identity

$$U(s)\bar U(s') \;=\; \delta(s - s') - \overrightarrow{\Delta}(s) f(s, s') \overleftarrow{\bar\Delta}(s') \;. \tag{5.8}$$

Note that all the fields considered above stem from D1-D1 strings. The remaining D1-D3 strings are responsible for imposing the BPS condition and the Nahm boundary conditions for the X^a at $s = \pm 1$.

The superextension of the Nahm construction is obtained, analogously to the superextension of the ADHM construction, by extending the Dirac operator (5.4) according to

$$\Delta^{\dot\alpha\dot\beta} \;=\; (\mathbb{1}_2)^{\dot\alpha\dot\beta} \otimes \frac{\partial}{\partial s} + \sigma_a^{(\dot\alpha\dot\beta)}(x^a - X^a) + (\eta_i^{(\dot\alpha}\chi^{\dot\beta)i}) \;. \tag{5.9}$$

The fields $\chi^{\dot\alpha i}$ are Weyl spinors and arise from the D1-D1 strings. (More explicitly, consider a bound state of D7-D5-branes, which dimensionally reduces to our D3-D1-brane system. The spinor $\chi^{\dot\alpha i}$ is the spinor χ^i_α we encountered before when discussing the $\mathcal{N} = (0, 1)$ hypermultiplet on the D5-brane.)

In the following, we will present a mapping to a configuration of topological D-branes, analogously to the one previously found for the ADHM construction.

5.2 The superspaces $\mathcal{Q}^{3|4}$ and $\hat{\mathcal{Q}}^{3|4}$

We want to consider a holomorphic Chern-Simons theory which describes magnetic monopoles and their superextensions. For this, we start from the holomorphic vector bundle

$$\mathcal{Q}^{3|4} \;=\; \mathcal{O}(2) \oplus \mathcal{O}(0) \oplus \mathbb{C}^4 \otimes \Pi\mathcal{O}(1) \tag{5.10}$$

of rank 2|4 over the Riemann sphere $\mathbb{C}P^1$. This bundle is covered by two patches $\tilde{\mathcal{V}}_\pm$ on which we have the coordinates $\lambda_\pm = w_2^\pm$ on the base space and $w_1^\pm, w_3^\pm$ in the bosonic fibres. On the overlap $\tilde{\mathcal{V}}_+ \cap \tilde{\mathcal{V}}_-$, we have thus[16]

$$w_+^1 \;=\; (w_+^2)^2 w_-^1 \;, \quad w_+^2 \;=\; \frac{1}{w_-^2} \;, \quad w_+^3 \;=\; w_-^3 \;. \tag{5.11}$$

The coordinates on the fermionic fibres of $\mathcal{Q}^{3|4}$ are the same as the ones on $\mathcal{P}^{3|4}$, i.e. we have $\eta_i^\pm$ with $i = 1, \dots 4$, satisfying $\eta_i^+ = \lambda_+ \eta_i^-$ on $\tilde{\mathcal{V}}_+ \cap \tilde{\mathcal{V}}_-$. From the Chern classes of the involved line bundles, we clearly see that $\mathcal{Q}^{3|4}$ is a Calabi-Yau supermanifold.

Note that holomorphic sections of the vector bundle $\mathcal{Q}^{3|4}$ are parameterized by elements $(y^{(\dot\alpha\dot\beta)}, y^4, \eta_i^{\dot\alpha})$ of the moduli space $\mathbb{C}^{4|8}$ according to

$$w_\pm^1 \;=\; y^{\dot\alpha\dot\beta}\lambda_{\dot\alpha}^\pm\lambda_{\dot\beta}^\pm \;, \quad w_\pm^3 \;=\; y^4 \;, \quad \eta_i^\pm \;=\; \eta_i^{\dot\alpha}\lambda_{\dot\alpha}^\pm \quad \text{with} \quad \lambda_\pm \;=\; w_\pm^2 \;. \tag{5.12}$$

Let us now deform and restrict the sections of $\mathcal{Q}^{3|4}$ by identifying y^4 with $-\gamma_\pm\lambda_{\dot\alpha}^\pm\hat\lambda_{\dot\beta}^\pm y^{\dot\alpha\dot\beta}$, where the coordinates $\hat\lambda_{\dot\alpha}$ were defined in (2.31). We still have $w_+^3 = w_-^3$ on the overlap $\tilde{\mathcal{V}}_+ \cap \tilde{\mathcal{V}}_-$, but w^3 no longer describes a section of a holomorphic line bundle. It is rather a section of a smooth line bundle, which we denote by $\hat{\mathcal{O}}(0)$. This deformation moreover reduces the moduli space from $\mathbb{C}^{4|8}$ to $\mathbb{C}^{3|8}$. We will denote the resulting total bundle by $\hat{\mathcal{Q}}^{3|4}$.

5.3 Field theories and dimensional reductions

First, we impose a reality condition on $\hat{\mathcal{Q}}^{3|4}$ which is (for the bosonic coordinates) given by

$$\tau(w_\pm^1, w_\pm^2) \;=\; \left(-\frac{\bar w_\pm^1}{(\bar w_\pm^2)^2}, -\frac{1}{\bar w_\pm^2}\right) \quad \text{and} \quad \tau(w_\pm^3) \;=\; \bar w_\pm^3 \tag{5.13}$$

and keep complex the coordinate $w_\pm^2$ on the base $\mathbb{C}P^1$, as usual. Then $w_\pm^1$ remains complex, but $w_\pm^3$ becomes real. In the identification with the real moduli $(x^1, x^3, x^4) \in \mathbb{R}^3$, we find that

$$y^{\dot1\dot1} \;=\; -(x^3 + \mathrm{i}x^4) \;=\; -\bar y^{\dot2\dot2} \quad \text{and} \quad w_\pm^3 \;=\; x^1 \;=\; -y^{\dot1\dot2} \;. \tag{5.14}$$

Thus, the space $\hat{\mathcal{Q}}^{3|4}$ reduces to a Cauchy-Riemann (CR) manifold[17], which we label by $\hat{\mathcal{Q}}_{-1}^{3|4} = \mathcal{K}^{5|8}$. This space has been extensively studied in [11], and it was found there that a partially holomorphic Chern-Simons theory obtained from a certain natural integrable

[16]The labelling of coordinates is chosen to become as consistent as possible with [11].

[17]Roughly speaking, a CR manifold is a complex manifold with additional real directions.

distribution on $\mathcal{K}^{5|8}$ is equivalent to the supersymmetric Bogomolny model on $\mathbb{R}^3$. Furthermore, it is evident that the complexification of this partially holomorphic Chern-Simons theory is holomorphic Chern-Simons theory on our space $\hat{\mathcal{Q}}^{3|4}$. This theory describes holomorphic structures $\bar{\partial}_{\mathcal{A}}$ on a vector bundle $\mathcal{E}$ over $\hat{\mathcal{Q}}^{3|4}$, i.e. a gauge potential $\mathcal{A}^{0,1}$ satisfying $\bar{\partial}\mathcal{A}^{0,1} + \mathcal{A}^{0,1} \wedge \mathcal{A}^{0,1} = 0$.

There are now three possibilities for (bosonic) dimensional reductions

$$\hat{\mathcal{Q}}^{3|4} \; = \; \mathcal{O}(2) \oplus \hat{\mathcal{O}}(0) \oplus \mathbb{C}^4 \otimes \Pi\mathcal{O}(1) \;\to\; \begin{cases} \mathcal{P}^{2|4} \;:=\; \mathcal{O}(2) \oplus \mathbb{C}^4 \otimes \Pi\mathcal{O}(1) \\ \hat{\mathcal{Q}}^{2|4} \;:=\; \hat{\mathcal{O}}(0) \oplus \mathbb{C}^4 \otimes \Pi\mathcal{O}(1) \\ \mathbb{C}P^{1|4} \;:=\; \mathbb{C}^4 \otimes \Pi\mathcal{O}(1) \end{cases} , \tag{5.15}$$

which we want to discuss next.

The dimensional reduction of the holomorphic Chern-Simons theory to the space $\mathcal{P}^{2|4}$ has also been studied in [11]. It yields a holomorphic BF-theory [64], where the scalar B-field originates as the component $\frac{\partial}{\partial \bar{w}^3_\pm} \lrcorner \mathcal{A}^{0,1}$ of the gauge potential $\mathcal{A}^{0,1}$ on $\mathcal{E} \to \hat{\mathcal{Q}}^{3|4}$. This theory is also equivalent to the above-mentioned super Bogomolny model on $\mathbb{R}^3$. It is furthermore the effective theory on a topological D3-brane and – via a Penrose-Ward transform – can be mapped to static BPS gauge configurations on a stack of D3-branes in type IIB superstring theory. These gauge configurations have been shown to amount to BPS D1-branes being suspended between the D3-branes and extending in their normal directions. Therefore, the holomorphic BF-theory is the topological analogue of the D3-brane point of view of the D3-D1-brane system.

From the above discussion, the field theory arising from the reduction to $\hat{\mathcal{Q}}^{2|4}$ is also evident. Note that considering this space is equivalent to considering $\hat{\mathcal{Q}}^{3|4}$ with the additional restriction $y^{\dot{1}\dot{1}} = y^{\dot{2}\dot{2}} = 0$. Therefore, we reduced the super Bogomolny model from $\mathbb{R}^3$ to $\mathbb{R}^1$, and we arrive at a (partially) holomorphic BF-theory, which is equivalent to self-dual Yang-Mills theory in one dimension. Since this theory yields precisely the gauge-covariant Nahm equations, we conclude that this is the D1-brane point of view of the D3-D1-brane system.

The last reduction proposed above is the one to $\mathbb{C}P^{1|4}$. This amounts to a reduction of the super Bogomolny model from $\mathbb{R}^3$ to a point, i.e. SDYM theory in zero dimensions. Thus, we arrive again at the matrix models (3.13) and (3.1) discussed previously. It is interesting to note that the matrix model cannot tell whether it originated from the space $\mathcal{P}^{3|4}$ or $\hat{\mathcal{Q}}^{3|4}$.

5.4 The Nahm construction from topological D-branes

In the previous section, we saw that both the physical D3-branes and the physical D1-branes correspond to topological D3-branes wrapping either the space $\mathcal{P}^{2|4} \subset \hat{\mathcal{Q}}^{3|4}$ or $\hat{\mathcal{Q}}^{2|4} \subset \hat{\mathcal{Q}}^{3|4}$. The bound system of D3-D1-branes therefore corresponds to a bound system of D3-D3-branes in the topological picture. The two D3-branes are separated by the same distance[18] as the physical ones in the normal direction $\mathcal{N}_{\mathcal{P}^{2|4}} \cong \mathcal{O}(2)$ in $\hat{\mathcal{Q}}^{3|4}$. It is important to stress,

[18] In our presentation of the Nahm construction, we chose this distance to be $1 - (-1) = 2$.

however, that since supersymmetry is broken twice by the D1- and the D3-branes, in the topological picture, we have to put to zero all fields except for $(A_a, \Phi, \chi^i_{\dot\alpha})$.

It remains to clarify the rôle of the Nahm boundary conditions. In [61], this was done by considering a D1-brane probe in a T-dualized configuration consisting of D7- and D5-branes. This picture cannot be translated easily into twistor space. It would be interesting to see explicitly what the boundary conditions correspond to in the topological setup. Furthermore, it could be enlightening to study the topological analogue of the Myers effect, which creates a funnel at the point where the physical D1-branes end on the physical D3-branes. Particularly the core of this "bion" might reveal interesting features in the topological theory.

6. Conclusions and outlook

In this paper, we presented various dimensional reductions of both holomorphic Chern-Simons theory on the supertwistor space $\mathcal{P}^{3|4}$ and the corresponding supersymmetric self-dual Yang-Mills theory on $\mathbb{R}^{4|8}$. In particular, we constructed two matrix models, one on $\mathbb{C}P^{1|4}$ and one on $\mathbb{R}^{0|8}$, whose solution spaces are bijective up to gauge transformations. We also defined similar matrix models by introducing noncommutativity on the moduli space of sections of the vector bundle $\mathcal{P}^{3|4}$ and treating both the supersymmetric self-dual Yang-Mills theory and the holomorphic Chern-Simons theory on the thus obtained deformed total space of $\mathcal{P}^{3|4}$ in the operator formalism.

Altogether, we obtained two matrix models on $\mathbb{C}P^{1|4}$, with actions closely related to $\mathcal{N}=2$ string field theory, and also two matrix models on $\mathbb{R}^{0|8}$.

We furthermore gave an interpretation of the matrix models in terms of D-brane configurations within B-type topological string theory. During this discussion, we established connections between topological branes and physical D-branes of type IIB superstring theory, whose worldvolume theory had been reduced by an additional BPS condition due to the presence of a further physical brane. Let us summarize the correspondences in the following table:

$$\begin{aligned}
\text{D}(5|4)\text{-branes in } \mathcal{P}^{3|4}_\varepsilon \;&\leftrightarrow\; \text{D}(3|8)\text{-branes in } \mathbb{R}^{4|8}\\
\text{D}(3|4)\text{-branes wrapping } \mathcal{P}^{2|4}_\varepsilon \text{ in } \mathcal{P}^{3|4}_\varepsilon \text{ or } \hat{\mathcal{Q}}^{3|4}_\varepsilon \;&\leftrightarrow\; \text{static D}(3|8)\text{-branes in } \mathbb{R}^{4|8}\\
\text{D}(3|4)\text{-branes wrapping } \hat{\mathcal{Q}}^{2|4}_\varepsilon \text{ in } \hat{\mathcal{Q}}^{3|4}_\varepsilon \;&\leftrightarrow\; \text{static D}(1|8)\text{-branes in } \mathbb{R}^{4|8}\\
\text{D}(1|4)\text{-branes in } \mathcal{P}^{3|4}_\varepsilon \;&\leftrightarrow\; \text{D}(\text{-}1|8)\text{-branes in } \mathbb{R}^{4|8}\ .
\end{aligned} \tag{6.1}$$

It should be stressed that the fermionic parts of all the branes in $\mathcal{P}^{3|4}_\varepsilon$ and $\hat{\mathcal{Q}}^{3|4}_\varepsilon$ only extend into holomorphic directions. It is straightforward to add to this list the diagonal line bundle $\mathcal{D}^{2|4}_\varepsilon$, which is obtained from $\mathcal{P}^{3|4}_\varepsilon$ by imposing a condition[19] $z^1_\pm = z^2_\pm$ on the local sections:

$$\text{D}(3|4)\text{-branes wrapping } \mathcal{D}^{2|4}_\varepsilon \text{ in } \mathcal{P}^{3|4}_\varepsilon \leftrightarrow \text{D}(1|8)\text{-branes in } \mathbb{R}^{4|8}\ . \tag{6.2}$$

[19] The condition for the Euclidean case is slightly different.

We furthermore established topological analogues of the D-brane configurations underlying both the super ADHM and the super Nahm construction. Thus, we found a matrix model over $\mathbb{C}P^{1|4}$ and a holomorphic BF-theory on $\hat{\mathcal{Q}}^{2|4}$ which are dual to holomorphic Chern-Simons theory on $\mathcal{P}^{3|4}$ and the holomorphic BF-theory on $\mathcal{P}^{2|4}$, respectively, in the same sense the ADHM and the Nahm equations are dual to the self-dual Yang-Mills and the Bogomolny equations, respectively.

From the results presented in this paper, there arise a number of interesting questions for further research. First, one should examine in more detail the topological D-brane configuration yielding the Nahm equations. In particular, it is desirable to obtain more results on the Myers effect and the core of the "bion" in the topological setting as already mentioned above. Second, one could imagine to strengthen and extend the relations between D-branes in type IIB superstring theory and the topological D-branes in the B-model. In the latter theory, the powerful framework of derived categories (see e.g. [65]) might then be carried over in some form to the full ten-dimensional string theory. Eventually, it might also be interesting to look at the mirror of the presented configurations in the topological A-model.

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